

GALOIS ACTIONS ON HOMOTOPY GROUPS OF ALGEBRAIC VARIETIES

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ABSTRACT. For any algebraic variety X defined over a finite or local field K , we study the Galois action on the Artin-Mazur homotopy groups $\pi_n^{\text{ét}}(X)$. If X is smooth, proper and of good reduction, with good fundamental group, we show that the action on $\pi_n^{\text{ét}}(X_{\bar{K}}) \otimes \mathbb{Q}_l$ is a mixed representation explicitly determined by the action on cohomology of Weil sheaves, whenever l is not equal to the residue characteristic p of K . A similar result for $l = p$ is then proved by comparing the crystalline and pro- p homotopy types.

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INTRODUCTION

In [AM], Artin and Mazur introduced the étale homotopy type of an algebraic variety. This gives rise to étale homotopy groups $\pi_n^{\text{ét}}(X, \bar{x})$; these are pro-finite groups, and $\pi_1^{\text{ét}}(X, \bar{x})$ is the usual étale fundamental group. In [Toë] §3.5.3, an approach for defining l -adic schematic homotopy types was discussed, giving l -adic schematic homotopy groups $\varpi_n(X, \bar{x})$; these are (pro-finite-dimensional) \mathbb{Q}_l -vector spaces when $n \geq 2$. In [Ols], Olsson introduced a crystalline schematic homotopy type, and established a comparison theorem with the p -adic schematic homotopy type.

Thus, given a variety X defined over a number field K , there are many notions of homotopy group: for each embedding $K \hookrightarrow \mathbb{C}$, the classical and schematic homotopy groups of the topological space $X_{\mathbb{C}}$; the étale homotopy groups of X ; the l -adic schematic homotopy groups of X ; over localisations $K_{\mathfrak{p}}$ of K , the crystalline schematic homotopy groups of $X_{K_{\mathfrak{p}}}$. However, despite their long heritage, very little was known even about the relation between étale and classical homotopy groups, unless the variety is simply connected. Our main aims are to establish comparisons between these various groups, and to study their structure and properties, especially the Galois actions induced on the étale and l -adic homotopy types.

In [Pri3], a new approach to studying non-abelian cohomology and schematic homotopy types of topological spaces was introduced. The bulk of this paper is concerned with adapting those techniques to pro-simplicial sets. This allows us to study Artin-Mazur homotopy types of algebraic varieties, and to establish arithmetic analogues of the results of [Pri4].

The main comparison results are Proposition 1.20 (showing when étale homotopy groups are profinite completions of classical homotopy groups), Theorem 2.44 (describing l -adic schematic homotopy groups in terms of étale homotopy groups), and Lemma 6.9 (comparing p -adic and crystalline homotopy groups). Theorem 5.10 and Corollary 5.15 show how to determine l -adic homotopy groups of smooth varieties over finite fields as Galois representations, by recovering them from cohomology groups of Weil sheaves. In particular, this implies that the l -adic homotopy groups are mixed Weil representations, thereby extending [Pri5] from fundamental groups to higher homotopy groups. Theorems 6.3 and 6.16 give similar results for l -adic and crystalline homotopy groups of varieties over local fields.

In Section 1, we recall standard definitions of pro-finite and pro- L homotopy types, and then establish some fundamental results. Proposition 1.13 shows how Kan's loop group can be used to construct the pro- L completion $X^{\hat{L}}$ of a space X , and Proposition 1.20 describes homotopy groups of $X^{\hat{L}}$.

We adapt these results in Section 2 to define non-abelian cohomology of a variety with coefficients in any simplicial algebraic group over \mathbb{Q}_l . It is then possible to apply the machinery developed in [Pri3], giving a non-nilpotent generalisation of the \mathbb{Q}_l -homotopy type of Weil II ([Del2]), and showing conditions under which we may recover étale homotopy groups from this (Theorem 2.44). Explicitly, if $\pi_1 X$ is algebraically good, and the higher homotopy groups have finite rank, then the higher pro-algebraic homotopy groups are just $\pi_n^{\text{ét}} X \otimes_{\mathbb{Z}} \mathbb{Q}_l$. For complex varieties, we also compare the pro-algebraic étale and analytic homotopy types.

Section 3 contains technical results showing how to extend the machinery of Section 2 to relative and filtered homotopy types. The former facilitate p -adic Hodge theory, while the latter are developed in order to study quasi-projective varieties. We also explore what it means for a pro-discrete group to act algebraically on a homotopy type. In Section 4, we investigate properties of homotopy types endowed with algebraic Galois actions.

In Section 5, the techniques of [Pri5] for studying Galois actions on algebraic groups then extend the finite characteristic results of [Pri1] to non-nilpotent and higher homotopy groups. The results are similar to [Pri4], substituting Frobenius actions for Hodge structures. Over finite fields, formality of the pro- \mathbb{Q}_l -algebraic homotopy type of a smooth projective variety (Theorem 5.10) follows from Lafforgue's Theorem and Deligne's Weil II theorems. For quasi-projective varieties, Corollary 5.15 establishes a property we call quasi-formality, analogous to Morgan's description of the rational homotopy type ([Mor]).

Over local fields in unequal characteristic, smooth specialisation suffices to adapt results from finite characteristic for varieties with good reduction. In equal characteristic, we show how pro- \mathbb{Q}_p -algebraic homotopy types relate to the framework of p -adic Hodge theory. Lemma 6.9 is a reworking of Olsson's non-abelian p -adic Hodge theory, and this has various consequences for Galois actions on Artin-Mazur homotopy groups (Theorems 6.13–6.16).

1. PRO-FINITE HOMOTOPY TYPES

Definition 1.1. Let \mathbb{S} be the category of simplicial sets, and take $s\mathbf{Gpd}$ to consist of those simplicial objects in the category of groupoids whose spaces of objects are discrete (i.e. sets, rather than simplicial sets).

Definition 1.2. Given a set L of primes, we say that a finite group G is an L -group if only primes in L divide its order. We define an L -groupoid to be a groupoid H for which $H(x, x)$ is an L -group for all $x \in \mathrm{Ob} H$.

Definition 1.3. Given a category \mathcal{C} , recall that the category $\mathrm{pro}\mathcal{C}$ of pro-objects in \mathcal{C} has objects consisting of filtered inverse systems $\{A_\alpha \in \mathcal{C}\}$, with

$$\mathrm{Hom}_{\mathcal{C}}(\{A_\alpha\}, \{B_\beta\}) = \varprojlim_{\beta} \varinjlim_{\alpha} \mathrm{Hom}_{\mathcal{C}}(A_\alpha, B_\beta).$$

Definition 1.4. Given a groupoid G and a set L of primes, define $G^{\hat{L}} \in \mathrm{pro}(\mathbf{Gpd})$ by requiring that $G^{\hat{L}}$ be the completion of G with respect to all L -groupoids H . In particular, $\mathrm{Ob} G^{\hat{L}} = \mathrm{Ob} G$ and $G^{\hat{L}}(x, x)$ is the pro- L completion of the group $G(x, x)$. If L is the set of all primes, we write $\hat{G} := G^{\hat{L}}$. Note that when G is a group, $G^{\hat{L}}$ is the pro- L completion in the sense of [Fri] §6, while \hat{G} is the profinite completion in the sense of [Ser2].

Given $G \in s\mathbf{Gpd}$, define $G^{\hat{L}}, \hat{G} \in \mathrm{pro}(s\mathbf{Gpd})$ by $(G^{\hat{L}})_n := (G_n)^{\hat{L}}, \hat{G}_n := \widehat{G_n}$.

Given $G = \{G_\alpha\} \in \mathrm{pro}(s\mathbf{Gpd})$ with discrete object set (i.e. $\mathrm{Ob} G \in \mathbf{Set} \subset \mathrm{pro}(\mathbf{Set})$), define $G^{\hat{L}} \in \mathrm{pro}(s\mathbf{Gpd})$ by $G^{\hat{L}} := \varprojlim (G_\alpha)^{\hat{L}}$.

Definition 1.5. As in [GJ] Ch.V.7, there is a classifying space functor $\bar{W} : s\mathbf{Gpd} \rightarrow \mathbb{S}$, with left adjoint $G : \mathbb{S} \rightarrow s\mathbf{Gpd}$, Dwyer and Kan's path groupoid functor ([DK]), and these give equivalences $\mathrm{Ho}(\mathbb{S}) \sim \mathrm{Ho}(s\mathbf{Gpd})$. The geometric realisation $|G(X)|$ is weakly equivalent to the path space of $|X|$. These functors have the additional

properties that $\text{Ob } G(X) = X_0$, $(\bar{W}G)_0 = \text{Ob } (G)$, $\pi_0 G(X) \cong \pi_0 |X|$, $\pi_0(|\bar{W}G|) \cong \pi_0 G_0$, $\pi_n(G(X)(x, x)) \cong \pi_{n+1}(|X|, x)$ and $\pi_{n+1}(|\bar{W}G|, x) = \pi_n(G(x, x))$. This allows us to study simplicial groupoids instead of topological spaces.

If $X \in \mathbb{S}$, then a local system is just a representation of the groupoid $\pi_f X$, i.e. a functor $\pi_f X \rightarrow \text{Gp}$ from the fundamental groupoid to the category of groups. As in [GJ] §VI.5, homotopy groups form a local system $\pi_n X$, whose stalk at x is $\pi_n(X, x)$. Given $X = \{X_\alpha\} \in \text{pro}(\mathbb{S})$, define the category of local systems on X to be the direct limit (over α) of the categories of local systems on X_α .

Definition 1.6. Given $X = \{X_\alpha\} \in \text{pro}(\mathbb{S})$ and a local system M on X_β define cohomology groups by

$$H^*(X, M) := \varinjlim_{\alpha} H^*(X_\alpha, M).$$

Given $G \in \text{pro}(s\text{Gpd})$, set $H^*(G, -) := H^*(\bar{W}G, -)$.

For X, M as above, define the cosimplicial complex $C^\bullet(X, M)$ by

$$C^n(X, M) := \text{Hom}_{\text{pro}(\text{Set})}(X_n, M),$$

noting that $H^*(C^\bullet(X, M)) = H^*(X, M)$.

Definition 1.7. Given $X \in \text{pro}(\mathbb{S})$ with X_0 discrete, and an inverse system $M = \{M_i\}_{i \in \mathbb{N}}$ of local systems on X , define the continuous cohomology groups $H^*(X, M)$ as follows (similarly to [Jan]). First form the cosimplicial complex $C^\bullet(X, M) = \varprojlim C^\bullet(X, M_i)$, for C^\bullet as in Definition 1.6, then set

$$H^*(X, M) := H^*(C^\bullet(X, M)).$$

Remark 1.8. Note that there is a short exact sequence

$$0 \rightarrow \varprojlim^1 H^{n-1}(X, M_i) \rightarrow H^n(X, M) \rightarrow \varprojlim H^n(X, M_i) \rightarrow 0,$$

so $H^n(X, M) \cong \varprojlim H^n(X, M_i)$ whenever the inverse system $\{H^{n-1}(X, M_i)\}_i$ satisfies the Mittag-Leffler condition (for instance if the groups are finite).

Lemma 1.9. Given $X \in \mathbb{S}$ and an inverse system $M = \{M_i\}_{i \in \mathbb{N}}$ of local systems on X , then

$$H^*(X, \varprojlim M_i) \cong H^*(X, M).$$

Proof. As in Definition 1.6, $H^*(X, \varprojlim M_i)$ is cohomology of the complex $C^\bullet(X, \varprojlim M_i)$, but

$$C^n(X, \varprojlim M_i) = \text{Hom}_{\text{Set}}(X_n, \varprojlim M_i) = \varprojlim \text{Hom}_{\text{Set}}(X_n, M_i) = C^n(X, M),$$

as required. \square

We will occasionally refer to groups and groupoids as “discrete”, to distinguish them from topological (or simplicial) groups and groupoids.

Definition 1.10. Given a set L of primes, say that a pro-discrete groupoid G with discrete object set is L -good if for all $(\pi_f G)^{\hat{L}}$ -representations M in abelian L -groups, the canonical map

$$\phi_M : H^*(G^{\hat{L}}, M) \rightarrow H^*(G, M)$$

is an isomorphism. When L is the set of all primes, we say that G is good. Observe that any inverse system of L -good groupoids is L -good.

Lemma 1.11. Free groups are L -good for all L .

Proof. Let $F = F(X)$ be a free group generated by a set X , and let $\Gamma := F^{\hat{L}}$. By the argument of [Ser2] I§2.6 Ex. 1(a), it suffices to show that $H^*(\Gamma, M) \rightarrow H^*(F, M)$ is surjective for all discrete Γ -representations M in abelian L -groups. Since F is free, $H^n(F, M) = 0$ for $n > 1$, so we need only verify that every derivation $\alpha : F \rightarrow M$ factors through Γ . The derivation gives rise to a map $\beta : F \rightarrow M \rtimes G$, for some finite L -torsion quotient G of F . Since $M \rtimes G$ is an L -group, β factors through Γ . \square

Examples 1.12. (1) All finite groups are L -good for all L .

(2) If $1 \rightarrow F \rightarrow \Gamma \rightarrow \Pi \rightarrow 1$ is an exact sequence of groups, with F and Π L -good and $H^a(F, M)$ finite for all finite L -torsion Γ -modules, then Γ is L -good.

(3) All finitely generated nilpotent groups are L -good for all L .

(4) The fundamental group of a compact Riemann surface is L -good for all L .

Proof. 2 This is essentially [Ser2] I§2.6 Ex. 2(c).

3 Express Γ as a successive extension of finite groups and \mathbb{Z} , then apply (2).

4 This amounts to showing that for a curve C of genus $g > 0$, the étale homotopy type $C_{\text{ét}}$ of C is a $K(\pi, 1)$. This is proved in [Sch] Proposition 15. \square

Proposition 1.13. *For any $X \in \mathbb{S}$, the canonical morphism*

$$X \rightarrow \bar{W}(G(X))^{\hat{L}}$$

in $\text{pro}(\mathbb{S})$ induces an isomorphism $(\pi_f X)^{\hat{L}} \rightarrow \pi_f(\bar{W}G(X))^{\hat{L}}$ of pro-groupoids, and has the property that for all finite abelian $(\pi_f X)^{\hat{L}}$ -representations M fibred in abelian L -groups, the canonical map

$$H^*(\bar{W}(G(X))^{\hat{L}}, M) \rightarrow H^*(X, M)$$

is an isomorphism.

Proof. The statement about fundamental groupoids is immediate, since completion commutes with taking quotients. Now, observe that

$$H^n(\bar{W}(G(X))^{\hat{L}}, M) \cong H^n(G(X))^{\hat{L}}, M).$$

It thus suffices to show that the simplicial groupoid $G(X)$ is L -good. This is equivalent to showing that for all $x \in X_0$, the simplicial groups $G(X)(x, x)$ are L -good. This will follow if the groups $G_n(x, x)$ are all L -good, since there is a spectral sequence

$$H^q(G_p, M) \implies H^{p+q}(G, M).$$

Since the groups $G_n(x, x)$ are all free, this then follows from Lemma 1.11. \square

For a property P of groups, we say that a Γ -representation H in groups locally satisfies P if the groups $H(x)$ all satisfy P .

Definition 1.14. Define $\mathcal{H} := \text{Ho}(\mathbb{S})$, and let $L\mathcal{H} \subset \mathcal{H}$ be the full subcategory consisting of spaces X for which the local systems $\pi_n(X)$ are locally L -groups. Let $\hat{\mathcal{H}} \subset \text{pro}(\mathcal{H})$ be the full subcategory consisting of spaces X with $\pi_0(X)$ discrete. Define $L\hat{\mathcal{H}}$ to be the full subcategory $L\hat{\mathcal{H}} := \text{pro}(L\mathcal{H}) \cap \hat{\mathcal{H}}$.

Proposition 1.15. *If $f : X \rightarrow Y$ is a morphism in $\hat{\mathcal{H}}$ such that $(\pi_f X)^{\hat{L}} \rightarrow \pi_f(Y)^{\hat{L}}$ is a pro-equivalence of pro-groupoids, and has the property that for all finite abelian $(\pi_f Y)^{\hat{L}}$ -representations M fibred in L -groups, the map*

$$H^*(f) : H^*(Y, M) \rightarrow H^*(X, M)$$

is an isomorphism, then for all $Z \in L\mathcal{H}$,

$$f^* : \text{Hom}_{\hat{\mathcal{H}}}(Y, Z) \rightarrow \text{Hom}_{\hat{\mathcal{H}}}(X, Z)$$

is an isomorphism.

Proof. Consider the Postnikov tower $P_n Z$ of Z . Assume that we have a homotopy class of maps $X \rightarrow P_n Z$. The obstruction to lifting this to a homotopy class of maps $X \rightarrow P_{n+1} Z$ lies in $H^{n+2}(X, \pi_{n+1} Z)$, and the latter homotopy class is a principal $H^{n+1}(X, \pi_{n+1} Z)$ -space or \emptyset . The isomorphism $H^*(Y, -) \cong H^*(X, -)$ means that a pro-homotopy class of morphisms $Y \rightarrow P_{n+1} Z$ is similarly determined, so the result follows by induction. \square

Corollary 1.16. *The inclusion functor $L\hat{\mathcal{H}} \rightarrow \hat{\mathcal{H}}$ has a left adjoint, which we denote by $X \rightsquigarrow X^{\hat{L}}$. Note that for $X \in L\hat{\mathcal{H}}$, $X^{\hat{L}} = X$.*

Proof. Propositions 1.13 and 1.15 imply that for $X \in \mathbb{S}$, $X^{\hat{L}} := \bar{W}(G(X))^{\hat{L}} \in L\hat{\mathcal{H}}$ has the required properties. Given an inverse system $X = \{X_\alpha\}$, set $X^{\hat{L}} := \varprojlim (X_\alpha)^{\hat{L}}$. \square

Remarks 1.17. Comparing with [Fri] Theorem 6.4 and Corollary 6.5, we see that this gives a generalisation of Artin and Mazur's pro- L homotopy type to unpointed spaces. If $X \in L\hat{\mathcal{H}}$, note that $X^{\hat{L}} = X$.

Since this paper was first written, an alternative pro-finite completion functor has been developed in [Qui1]. However, this differs in that it also takes the pro-finite completion of $\pi_0 X$.

1.1. Comparing homotopy groups. We now investigate when we can describe the homotopy groups of $X^{\hat{L}}$ in terms of the homotopy groups of X .

Lemma 1.18. *If A is a finitely generated abelian group, then for $n \geq 2$, completion of the Eilenberg-MacLane space is given by $K(A, n)^{\hat{L}} = K(A^{\hat{L}}, n)$.*

Proof. By Proposition 1.15, we need to show that the maps

$$H^*(K(A^{\hat{L}}, n), M) \rightarrow H^*(K(A, n), M)$$

are isomorphisms for all abelian L -groups M . By considering the spectral sequence associated to a filtration, it suffices to consider only the cases $M = \mathbb{F}_p$, for $p \in L$.

If $A = A' \times A''$, then $K(A, n) = K(A', n) \times K(A'', n)$, so $H^*(K(A, n), \mathbb{F}_p) = H^*(K(A', n), \mathbb{F}_p) \otimes H^*(K(A'', n), \mathbb{F}_p)$. The structure theorem for finitely generated abelian groups therefore allows us to assume that $A = \mathbb{Z}/q$, for q a prime power or 0.

Now, if q is neither zero nor a power of p , then $H^r(K(A, n), \mathbb{F}_p) = 0$ for $r > 0$; since $A^{\hat{L}}$ is a quotient of A , we also get $H^r(K(A^{\hat{L}}, n), \mathbb{F}_p) = 0$. If $q = p^s$, then $A^{\hat{L}} = A$, making isomorphism automatic.

If $q = 0$, then $A = \mathbb{Z}$, $A^{\hat{L}} = \prod_{l \in L} \mathbb{Z}_l$, and $H^r(K(\mathbb{Z}_l, n), \mathbb{F}_p) = 0$ for $r > 0$ and $l \neq p$. We need to show that

$$H^*(K(\mathbb{Z}_p, n), \mathbb{F}_p) \rightarrow H^*(K(\mathbb{Z}, n), \mathbb{F}_p)$$

is an isomorphism, or equivalently that $K(\mathbb{Z}, n)^{\hat{p}} = K(\mathbb{Z}_p, n)$. This follows from [Qui3] Theorem 1.5. \square

Definition 1.19. Given a group-valued representation H of a groupoid Γ , recall from [Pri3] Definition 2.15 that the semi-direct product $H \rtimes \Gamma$ is a groupoid with objects $\text{Ob}(H \rtimes \Gamma) = \text{Ob}(\Gamma)$ and has $(H \rtimes \Gamma)(x, x) = H_x \rtimes \Gamma(x, x)$.

Given a property P of groups, we will say that a groupoid Γ locally satisfies P if the groups $\Gamma(x, x)$ satisfy P , for all $x \in \text{Ob} \Gamma$.

Proposition 1.20. *Fix $X \in \mathbb{S}$. If $\pi_f X$ is L -good, with $\pi_n(X) \otimes \mathbb{F}_p$ locally finite-dimensional and if the image of $\pi_1(X) \rightarrow \text{Aut}(\pi_n(X) \otimes \mathbb{F}_p)$ is L -torsion for all n and all $p \in L$, then the natural maps*

$$\pi_n(X)^{\hat{L}} \rightarrow \pi_n(X^{\hat{L}})$$

are isomorphisms for all n .

Proof. We adapt the argument of [Pri3] Theorem 1.58. Let $X(n)$ be the Postnikov tower for X . We will prove the proposition inductively for the groups $X(n)$. Write $\Gamma := \pi_f X$.

For $n = 1$, consider the map $G(X(1))^{\hat{L}} \rightarrow \Gamma^{\hat{L}}$. For all finite L -torsion representations M of $\Gamma^{\hat{L}}$, this induces isomorphisms

$$H^*(\Gamma^{\hat{L}}, M) \rightarrow H^*(\Gamma, M) \cong H^*(G(X(1)), M)$$

on cohomology (by goodness), so Proposition 1.15 implies that it is a weak equivalence.

Now assume that $X(n-1)$ satisfies the inductive hypothesis, and consider the fibration $X(n) \rightarrow X(n-1)$. This is determined up to homotopy by a k -invariant ([GJ] §VI.5) $\kappa \in H^{n+1}(X(n-1), \pi_n(X))$. Since $\pi_n(X) \otimes \mathbb{F}_p$ is a finite-dimensional $\Gamma^{\hat{L}}$ -representation for all $p \in L$, the group $A := \pi_n(X)^{\hat{L}}$ is an inverse limit of finite $\Gamma^{\hat{L}}$ -representations. Now, the element

$$\kappa \in H^{n+1}(X(n-1), A) \cong H^{n+1}(X(n-1)^{\hat{L}}, A)$$

comes from a map

$$G(X(n-1))^{\hat{L}} \rightarrow (N^{-1}A[-n]) \rtimes \Gamma,$$

where N^{-1} denotes the denormalisation functor from chain complexes to simplicial complexes (the Dold-Kan correspondence).

Let LA be the chain complex with A concentrated in degrees $n, n-1$, and $d : (LA)_n \rightarrow (LA)_{n-1}$ the identity, and define \mathcal{G} to be the pullback of this map along the surjection $N^{-1}L \rtimes \Gamma \rightarrow (N^{-1}A[-n]) \rtimes \Gamma$ of simplicial locally profinite L -torsion groupoids. This gives an extension

$$N^{-1}A[1-n] \rightarrow \mathcal{G} \rightarrow G(X(n-1))^{\hat{L}}.$$

Applying \bar{W} gives the fibration

$$\bar{W}N^{-1}A[1-n] \rightarrow \bar{W}\mathcal{G} \rightarrow X(n-1)^{\hat{L}}$$

in \mathbb{S} , corresponding to the k -invariant $f^*\kappa \in H^n(X(n-1)^{\hat{L}}, A)$ for $f : X(n-1) \rightarrow X(n-1)^{\hat{L}}$. This in turn gives a map $X(n) \rightarrow \bar{W}\mathcal{G}$, compatible with the fibrations.

From the long exact sequence of homotopy, it follows that \mathcal{G} has the required homotopy groups, and $\bar{W}\mathcal{G} \in L\mathcal{H}$. It will therefore suffice to show that $F : G(X(n))^{\hat{L}} \rightarrow \mathcal{G}$ is a weak equivalence. We now apply the Hochschild-Serre spectral sequence, giving

$$H^p(X(n-1), H^q(N^{-1}A[1-n], M)) = H^p(G(X(n-1))^{\hat{L}}, H^q(N^{-1}A[1-n], M)) \implies H^{p+q}(\mathcal{G}, M).$$

Similarly

$$H^p(X(n-1), H^q(E(n), V)) \implies H^{p+q}(X(n), V),$$

for all $\Gamma^{\hat{L}}$ -representations M in abelian L -groups, where $E(n)$ is the fibre of $X(n) \rightarrow X(n-1)$.

Now, $E(n)$ is a $K(\pi_n(X), n)$ -space, and $\bar{W}N^{-1}A[1-n]$ is a $K(A, n)$ -space. By Lemma 1.18, it follows that $E(n) \rightarrow \bar{W}N^{-1}A[1-n]$ is pro- L localisation, giving an isomorphism of cohomology with coefficients in M . Thus F induces isomorphisms on homology groups, hence must be a weak equivalence by Proposition 1.15. \square

2. PRO- \mathbb{Q}_l -ALGEBRAIC HOMOTOPY TYPES

Fix a prime l . Although all results here will be stated for the local field \mathbb{Q}_l , they hold for any of its algebraic extensions. Throughout this section we will retain the notation and conventions of [Pri3] concerning pro-algebraic groupoids.

2.1. Revision of pro-algebraic groupoids. We first recall some definitions from [Pri3] §§2.1–2.3.

Definition 2.1. Define a pro-algebraic groupoid G over k to consist of the following data:

- (1) A discrete set $\text{Ob}(G)$.
- (2) For all $x, y \in \text{Ob}(G)$, an affine scheme $G(x, y)$ (possibly empty) over k .
- (3) A groupoid structure on G , consisting of an associative multiplication morphism $m : G(x, y) \times G(y, z) \rightarrow G(x, z)$, identities $\text{Spec } k \rightarrow G(x, x)$ and inverses $G(x, y) \rightarrow G(y, x)$

Note that a pro-algebraic group is just a pro-algebraic groupoid on one object. We say that a pro-algebraic groupoid is reductive (resp. pro-unipotent) if the pro-algebraic groups $G(x, x)$ are so for all $x \in \text{Ob}(G)$. An algebraic groupoid is a pro-algebraic groupoid for which the $G(x, y)$ are all of finite type.

If G is a pro-algebraic groupoid, let $O(G(x, y))$ denote the global sections of the structure sheaf of $G(x, y)$.

Definition 2.2. Given morphisms $f, g : G \rightarrow H$ of pro-algebraic groupoids, define a natural isomorphism η between f and g to consist of morphisms

$$\eta_x : \text{Spec } k \rightarrow H(f(x), g(x))$$

for all $x \in \text{Ob}(G)$, such that the following diagram commutes, for all $x, y \in \text{Ob}(G)$:

$$\begin{array}{ccc} G(x, y) & \xrightarrow{f(x, y)} & H(f(x), f(y)) \\ g(x, y) \downarrow & & \downarrow \cdot \eta_y \\ H(g(x), g(y)) & \xrightarrow{\eta_x \cdot} & H(f(x), g(y)). \end{array}$$

A morphism $f : G \rightarrow H$ of pro-algebraic groupoids is said to be an equivalence if there exists a morphism $g : H \rightarrow G$ such that fg and gf are both naturally isomorphic to identity morphisms. This is the same as saying that for all $y \in \text{Ob}(H)$, there exists $x \in \text{Ob}(G)$ such that $H(f(x), y)(k)$ is non-empty (essential surjectivity), and that for all $x_1, x_2 \in \text{Ob}(G)$, $G(x, y) \rightarrow G(f(x_1), f(x_2))$ is an isomorphism.

Definition 2.3. Given a pro-algebraic groupoid G , define a finite-dimensional linear G -representation to be a functor $\rho : G \rightarrow \text{FDVect}_k$ respecting the algebraic structure. Explicitly, this consists of a set $\{V_x\}_{x \in \text{Ob}(G)}$ of finite-dimensional k -vector spaces, together with morphisms $\rho_{xy} : G(x, y) \rightarrow \text{Hom}(V_y, V_x)$ of affine schemes, respecting the multiplication and identities.

A morphism $f : (V, \rho) \rightarrow (W, \varrho)$ of G -representations consists of $f_x \in \text{Hom}(V_x, W_x)$ such that

$$f_x \circ \varrho_{xy} = \rho_{xy} \circ f_y : G(x, y) \rightarrow \text{Hom}(V_x, W_y).$$

Definition 2.4. Given a pro-algebraic groupoid G , define the reductive quotient G^{red} of G by setting $\text{Ob}(G^{\text{red}}) = \text{Ob}(G)$, and

$$G^{\text{red}}(x, y) = G(x, y) / R_u(G(y, y)) = R_u(G(x, x)) \backslash G(x, y),$$

where $R_u(G(x, x))$ is the pro-unipotent radical of the pro-algebraic group $G(x, x)$. The equality arises since if $f \in G(x, y)$, $g \in R_u(G(y, y))$, then $f g f^{-1} \in R_u(G(x, x))$, so both equivalence relations are the same. Multiplication and inversion descend similarly. Observe that G^{red} is then a reductive pro-algebraic groupoid. Representations of G^{red} correspond to semisimple representations of G .

Definition 2.5. Let AGpd denote the category of pro-algebraic groupoids over k , and observe that this category contains all (inverse) limits. There is functor from AGpd to Gpd , the category of abstract groupoids, given by $G \mapsto G(k)$. This functor preserves all limits, so has a left adjoint, the algebraisation functor, denoted $\Gamma \mapsto \Gamma^{\text{alg}}$. This can be given explicitly by $\text{Ob}(\Gamma)^{\text{alg}} = \text{Ob}(\Gamma)$, and

$$\Gamma^{\text{alg}}(x, y) = \Gamma(x, x)^{\text{alg}} \times^{\Gamma(x, x)} \Gamma(x, y),$$

where $\Gamma(x, x)^{\text{alg}}$ is the pro-algebraic completion of the group $\Gamma(x, x)$.

The finite-dimensional linear representations of Γ (as in Definition 2.3) correspond to those of Γ^{alg} , and these can be used to recover Γ^{alg} , by Tannakian duality.

Definition 2.6. Given a pro-algebraic groupoid G , and $U = \{U_x\}_{x \in \text{Ob}(G)}$ a collection of pro-algebraic groups parametrised by $\text{Ob}(G)$, we say that G acts on U if there are morphisms $U_x \times G(x, y) \xrightarrow{*} U_y$ of affine schemes, satisfying the following conditions:

- (1) $(uv) * g = (u * g)(v * g)$, $1 * g = 1$ and $(u^{-1}) * g = (u * g)^{-1}$, for $g \in G(x, y)$ and $u, v \in U_x$.
- (2) $u * (gh) = (u * g) * h$ and $u * 1 = u$, for $g \in G(x, y)$, $h \in G(y, z)$ and $u \in U_x$.

If G acts on U , we write $G \ltimes U$ for the groupoid given by

- (1) $\text{Ob}(G \ltimes U) := \text{Ob}(G)$.
- (2) $(G \ltimes U)(x, y) := G(x, y) \times U_y$.
- (3) $(g, u)(h, v) := (gh, (u * h)v)$ for $g \in G(x, y)$, $h \in G(y, z)$ and $u \in U_y, v \in U_z$.

Definition 2.7. Given a pro-algebraic groupoid G , define $R_u(G)$ to be the collection $R_u(G)_x = R_u(G(x, x))$ of pro-unipotent pro-algebraic groups, for $x \in \text{Ob}(G)$. G then acts on $R_u(G)$ by conjugation, i.e.

$$u * g := g^{-1} u g,$$

for $u \in R_u(G)_x$, $g \in G(x, y)$.

Proposition 2.8. *For any pro-algebraic groupoid G , there is a Levi decomposition $G = G^{\text{red}} \ltimes R_u(G)$, unique up to conjugation by $R_u(G)$.*

Proof. [Pri3] Proposition 2.17. □

2.2. Algebraisation of locally profinite groupoids.

Definition 2.9. Given a pro-discrete groupoid Γ with $\text{Ob}(\Gamma)$ a discrete set, we define the pro-algebraic completion Γ^{alg} to be the pro- \mathbb{Q}_l -algebraic groupoid pro-representing the functor

$$\begin{aligned} \text{AGpd} &\rightarrow \text{Set} \\ H &\mapsto \text{Hom}_{\text{TopGpd}}(\Gamma, H), \end{aligned}$$

where TopGpd denotes the category of topological groupoids. Note that this exists by the Special Adjoint Functor Theorem ([Mac] Theorem V.8.2). Given a set of primes L , define the L -algebraic completion $\Gamma^{L,\text{alg}}$ to be $(\Gamma^{\hat{L}})^{\text{alg}}$. If P is the set of all primes, we simply write $\hat{\Gamma}^{\text{alg}} := \Gamma^{P,\text{alg}}$.

Remark 2.10. Since representations with finite monodromy are algebraic there is a canonical retraction $\Gamma^{L,\text{alg}} \rightarrow \Gamma^{\hat{L}}$ of pro-algebraic groupoids.

We adapt the following definition from [Hai] to pro-finite groups:

Definition 2.11. Given a pro-groupoid Γ with $\text{Ob}(\Gamma)$ discrete, a reductive pro-algebraic groupoid R over \mathbb{Q}_l , and a Zariski-dense (i.e. essentially surjective on objects and Zariski-dense on morphisms) continuous map

$$\rho : \Gamma^{\hat{L}} \rightarrow R(\mathbb{Q}_l),$$

where the latter is given the l -adic topology, we define the relative Malcev completion $\Gamma^{L,\rho,\text{Mal}}$ to be the universal diagram

$$\Gamma^{\hat{L}} \xrightarrow{g} \Gamma^{L,\rho,\text{Mal}}(\mathbb{Q}_l) \xrightarrow{f} R(\mathbb{Q}_l),$$

with $f : \Gamma^{L,\rho,\text{Mal}} \rightarrow R$ a pro-unipotent extension of pro- \mathbb{Q}_l -algebraic groupoids, g a continuous map of topological groupoids, and the composition equal to ρ .

To see that this universal object exists, we note that this description determines the linear representations of $\Gamma^{L,\rho,\text{Mal}}$, as described in Remarks 2.12. Since these form a multi-fibred tensor category, Tannakian duality ([Pri3] Remark 2.6) then gives a construction of $\Gamma^{L,\rho,\text{Mal}}$.

Remarks 2.12. Observe that finite-dimensional linear representations of $\Gamma^{L,\text{alg}}$ are continuous \mathbb{Q}_l -representations of $\Gamma^{\hat{L}}$. Finite-dimensional representations of $\Gamma^{L,\rho,\text{Mal}}$ are only those continuous \mathbb{Q}_l -representations whose semisimplifications are R -representations. Moreover, if we let R be the reductive quotient $\Gamma^{L,\text{red}}$ of $\Gamma^{L,\text{alg}}$, then $\Gamma^{L,\text{alg}} = \Gamma^{L,\rho,\text{Mal}}$.

Lemma 2.13. *If Γ is a pro-finite group, V an n -dimensional \mathbb{Q}_l -vector space, and $\rho : \Gamma \rightarrow \text{GL}(V)$ a continuous representation (where the latter is given the l -adic topology) then there exists a lattice (i.e. a rank n \mathbb{Z}_l -submodule $\Lambda \subset V$) such that ρ factors through $\text{GL}(\Lambda)$.*

Proof. Since Γ is pro-finite, it is compact, and hence $\rho(\Gamma) \leq \text{GL}(V)$ must be compact. [Ser1] LG 4 Appendix 1 Theorems 1 and 2 show that every compact subgroup of $\text{GL}(V)$ is contained in a maximal compact subgroup, and that the maximal compact subgroups are of the form $\text{GL}(\Lambda)$. \square

Proposition 2.14. *Given a locally profinite groupoid Γ with discrete objects, and a Zariski-dense continuous map*

$$\rho : \pi_f(\Gamma)^{\hat{L}} \rightarrow G(\mathbb{Q}_l)$$

to a pro- \mathbb{Q}_l -algebraic groupoid, there is a model $G_{\mathbb{Z}_l}$ for G over \mathbb{Z}_l , unique subject to the property that ρ factors through a Zariski-dense map

$$\rho_{\mathbb{Z}_l} : \pi_f(\Gamma)^{\hat{L}} \rightarrow G_{\mathbb{Z}_l}(\mathbb{Z}_l).$$

Proof. Assume that ρ is an isomorphism on objects (replacing G by an equivalent groupoid). Let \mathcal{C} be the category of continuous Γ -representations in finite free \mathbb{Z}_l -modules. For each $x \in \text{Ob } \Gamma$, this gives a fibre functor ω_x from \mathcal{C} to finite free \mathbb{Z}_l -modules.

If we let \mathcal{D} be the category of Γ -representations in finite-dimensional \mathbb{Q}_l -vector spaces, with the fibre functors also denoted by ω_x , then the category of G -representations is equivalent to a full subcategory $\mathcal{D}(G)$ of \mathcal{D} , since ρ is Zariski-dense. By Tannakian duality (as in [Pri3] §2.1), there are isomorphisms

$$G(x, y)(A) \cong \text{Iso}^{\otimes}(\omega_x|_{\mathcal{D}(G)}, \omega_y|_{\mathcal{D}(G)})(A).$$

Now, by Lemma 2.13, the functor $\otimes \mathbb{Q}_l : \mathcal{C} \rightarrow \mathcal{D}$ is essentially surjective. Let $\mathcal{C}(G)$ be the full subcategory of \mathcal{C} whose objects are those Λ for which $\Lambda \otimes \mathbb{Q}_l$ is isomorphic to an object of $\mathcal{D}(G)$; these are Γ -lattices in G -representations. Define

$$G_{\mathbb{Z}_l}(x, y)(A) := \text{Iso}^{\otimes}(\omega_x|_{\mathcal{C}(G)}, \omega_y|_{\mathcal{C}(G)})(A),$$

observing that this is an affine scheme (since it preserves all inverse limits), with $G_{\mathbb{Z}_l} \otimes \mathbb{Q}_l = G$. \square

Definition 2.15. Given a finite-dimensional nilpotent Lie algebra \mathfrak{u} over \mathbb{Q}_l , equipped with the continuous action of a profinite group Γ (respecting the Lie algebra structure), we say that a lattice $\Lambda \subset \mathfrak{u}$ is admissible if it satisfies the following:

- (1) Λ is a Γ -subrepresentation;
- (2) Λ is closed under all the operations in the Campbell-Baker-Hausdorff formula for $\log(e^a \cdot e^b)$ (i.e. $\frac{1}{2}[a, b]$, $\frac{1}{12}([a, [a, b]] - [b, [a, b]])$, $\frac{1}{24}[a, [b, [a, b]]]$, ...).

Definition 2.16. Let $\mathcal{N}_{\mathbb{Q}_l}$ be the category of finite-dimensional nilpotent \mathbb{Q}_l -Lie algebras. For any pro-algebraic groupoid R over \mathbb{Q}_l , let $\mathcal{N}(R)$ be the category of R -representations in $\mathcal{N}_{\mathbb{Q}_l}$. Write $\hat{\mathcal{N}}(R)$ for the category of pro-objects of $\mathcal{N}(R)$, and $s\hat{\mathcal{N}}(R)$ for the category of simplicial objects in $\hat{\mathcal{N}}(R)$.

Lemma 2.17. *If $\Lambda \subset \mathfrak{u}$ is an admissible lattice and $\mathfrak{u} \in \mathcal{N}$, then the image of Λ under the exponential map*

$$\exp : \mathfrak{u} \rightarrow \exp(\mathfrak{u})$$

is a profinite subgroup.

Proof. We may regard $\exp(\mathfrak{u})$ as being the set \mathfrak{u} , with multiplication given by the Campbell-Baker-Hausdorff formula. Since Λ is closed under all the operations in the formula, it is closed under multiplication. As \exp is a homeomorphism, $\exp(\Lambda)$ is compact and thus profinite. \square

2.3. Pro- \mathbb{Q}_l -algebraic homotopy types.

Definition 2.18. Given a pro-simplicial groupoid G with $\text{Ob}(G)$ a discrete set, we define the pro-algebraic completion $G^{L, \text{alg}} \in s\text{AGpd}$ to represent the functor

$$\begin{aligned} s\text{AGpd} &\rightarrow \text{Set} \\ H &\mapsto \text{Hom}_{s\text{TopGpd}}(G^{\hat{L}}, H), \end{aligned}$$

where TopGpd denotes the category of topological groupoids. Note that this exists by the Special Adjoint Functor Theorem ([Mac] Theorem V.8.2).

Definition 2.19. Given a pro-simplicial groupoid G with $\text{Ob}(G)$ discrete, a reductive pro-algebraic groupoid R over \mathbb{Q}_l , and a Zariski-dense continuous map

$$\rho : \pi_f(G)^{\hat{L}} \rightarrow R(\mathbb{Q}_l),$$

where the latter is given the l -adic topology, we define the relative Malcev completion $G^{L,\rho,\text{Mal}} \in \text{sAGpd} \downarrow R$ by $(G^{L,\rho,\text{Mal}})_n := (G_n)^{L,\rho \circ a_n, \text{Mal}}$, for $a_n : G_n \rightarrow \pi_f G$ the canonical map.

Note that $\pi_f(G^{L,\rho,\text{Mal}}) = \pi_f(G)^{L,\rho,\text{Mal}}$.

Lemma 2.20. *If the continuous action of a profinite group Γ on $\mathfrak{u}_\bullet \in \text{s}\mathcal{N}_{\mathbb{Q}_l}$ is semisimple, then \mathfrak{u} is the union of its simplicial admissible sublattices.*

Proof. Since the action of Γ is semisimple, we may take a complement $V_\bullet \subset \mathfrak{u}_\bullet$ of $[\mathfrak{u}_\bullet, \mathfrak{u}_\bullet]$ as a simplicial Γ -representation. Given a lattice $M \subset V$, let $g(M) \subset \mathfrak{u}$ denote the \mathbb{Z}_l -submodule generated by M and the operations in the Campbell-Baker-Hausdorff formula. Since \mathfrak{u} is nilpotent, it follows that $g(M)$ is a finitely generated \mathbb{Z}_l -module, and hence a lattice in \mathfrak{u} . By semisimplicity and Lemma 2.13, there exists a Γ -equivariant lattice $\Lambda_\bullet \subset V_\bullet$. The lattices $l^{-n}\Lambda_\bullet \subset V_\bullet$ are also then Γ -equivariant for $n \geq 0$, so the lattices $g(l^{-n}\Lambda_\bullet) \subset \mathfrak{u}_\bullet$ are all admissible.

It only remains to show that $\bigcup g(l^{-n}\Lambda) \rightarrow \mathfrak{u}$ is a surjective map of Lie algebras. This follows since $\bigcup l^{-n}\Lambda \rightarrow \mathfrak{u}/[\mathfrak{u}, \mathfrak{u}]$ is surjective. \square

Lemma 2.21. *Given a compact topological Γ -space K and a finite-dimensional nilpotent \mathbb{Q}_l -Lie algebra, the map*

$$\text{Hom}_{\text{cts}}(K, \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathfrak{u} \rightarrow \text{Hom}_{\text{cts}}(K, \mathfrak{u})$$

is an isomorphism.

Proof. First observe that the map is clearly injective, since \mathfrak{u} is a flat \mathbb{Z}_l -module. For surjectivity, note that the image of $f : K \rightarrow \mathfrak{u}$ must be contained in an admissible sublattice $\Lambda \subset \mathfrak{u}$ (by compactness and Lemma 2.20). Now,

$$\text{Hom}_{\text{cts}}(K, \Lambda) \cong \text{Hom}_{\text{cts}}(K, \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \Lambda,$$

since Λ is a free \mathbb{Z}_l -module. \square

Definition 2.22. Given a representation V of $\widehat{\pi_f X}$ in \mathbb{Q}_l -vector spaces, such that the map $\pi_1(X) \rightarrow \text{GL}(V)$ is continuous (when $\text{GL}(V)$ is given the l -adic topology), recall that we define

$$H^*(X, V) := H^*(X, \Lambda) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l,$$

for any $\pi_f X$ -equivariant \mathbb{Z}_l -lattice $\Lambda \subset V$ as in Lemma 2.13, and $H^*(X, \Lambda)$ as in Definition 1.7.

Remark 2.23. If X is discrete, note that this is not in general the same as cohomology $H^n(X, V^\delta)$ of the discrete $\pi_f G$ -representation V^δ underlying V . However, both will coincide if $H_n(G, \Lambda^\vee)$ has finite rank, by the Universal Coefficient Theorem and Lemma 1.9.

Example 2.24. If X is a locally Noetherian simplicial scheme, we may consider the étale topological type $X_{\text{ét}} \in \text{pro}(\mathbb{S})$, as defined in [Fri] Definition 4.4. Since $(X_{\text{ét}})_0$ is the set of geometric points of X_0 , we may then apply the constructions of this section. For a finite local system M on X , we have

$$H^*(X_{\text{ét}}, M) \cong H_{\text{ét}}^*(X, M),$$

by [Fri] Proposition 5.9. For an inverse system $M = \{M_i\}$ of local systems, we have

$$H^*(X_{\text{ét}}, M) = H^*(\varprojlim_i C_{\text{ét}}^\bullet(X, M_i)) = H_{\text{ét}}^*(X, (M)),$$

where $C_{\text{ét}}^\bullet$ is a variant of the Godement resolution and $H_{\text{ét}}^*(X, (M))$ is Jannsen's continuous étale cohomology ([Jan]). If the groups $H_{\text{ét}}^*(X, M_i)$ satisfy the Mittag-Leffler condition (in particular, if they are finite), then

$$H^*(X_{\text{ét}}, M) \cong \varprojlim_i H_{\text{ét}}^*(X, M_i).$$

[Fri] Theorem 7.3 shows that $X_{\text{ét}} \in \hat{\mathcal{H}}$ whenever the schemes X_n are connected and geometrically unbranched. It seems that this result can be extended to simplicial spaces for which the homotopy groups $\pi_m^{\text{ét}}(X_n)$ satisfy the π_* -Kan condition ([GJ] §IV.4), provided the simplicial set $\pi(X)_\bullet$, given by $\pi(X)_n := \pi(X_n)$, the set of connected components of X_n , has finite homotopy groups.

Proposition 2.25. *Take $X \in \text{pro}(\mathbb{S})$ with X_0 discrete, and a Zariski-dense continuous map*

$$\rho : \pi_f(X)^{\hat{L}} \rightarrow R(\mathbb{Q}_l),$$

for $l \in L$. Then $G(X)^{L, \rho, \text{Mal}}$ is cofibrant, $G(X)^{L', \rho, \text{Mal}} \rightarrow G(X)^{L, \rho, \text{Mal}}$ is an isomorphism for all $L \subset L'$, and

$$H^*(G(X)^{L, \rho, \text{Mal}}, V) \cong H^*(X, \rho^*V).$$

Proof. Let $\Delta \leq R(\mathbb{Q}_l)$ be the image of ρ . Write $\{X_\alpha\}_{\alpha \in \mathbb{I}}$ for the inverse system X . For $\mathbf{u} \in s\mathcal{N}(R)$,

$$\begin{aligned} \text{Hom}_{s\text{TopGpd}}(G(X)^{\hat{L}}, \exp(\mathbf{u}) \rtimes R)_R &= \text{Hom}_{s\text{TopGpd}}(\mathcal{G}(X)^{\hat{L}}, \exp(\mathbf{u}) \rtimes \Delta)_\Delta \\ &= \varinjlim_{\Lambda \subset \mathbf{u} \text{ admissible}} \text{Hom}_{s\text{TopGpd}}(G(X)^{\hat{L}}, \exp(\Lambda) \rtimes \Delta)_\Delta \\ &= \varinjlim_{\Lambda \subset \mathbf{u} \text{ admissible}} \text{Hom}_{\text{pro}(s\text{Gpd})}(G(X), \exp(\Lambda) \rtimes \Delta)_\Delta, \\ &= \varinjlim_{\Lambda \subset \mathbf{u} \text{ admissible}} \text{Hom}_{\text{pro}(\mathbb{S})}(X, \bar{W}(\exp(\Lambda) \rtimes \Delta))_{\bar{W}\Delta}, \\ &= \varinjlim_{\Lambda \subset \mathbf{u} \text{ admissible}} \varprojlim_n \varinjlim_\alpha \text{Hom}_{\mathbb{S}}(X_\alpha, \bar{W}(\exp(\Lambda/l^n) \rtimes \Delta))_{\bar{W}\Delta}. \end{aligned}$$

This expression is independent of L , so we have shown that $G(X)^{L', \rho, \text{Mal}} \rightarrow G(X)^{L, \rho, \text{Mal}}$ is an isomorphism for all $L \subset L'$.

For $p : \mathbf{u} \rightarrow \mathbf{v}$ an acyclic small extension with kernel I in $s\mathcal{N}(R)$, and an admissible lattice $\Lambda' < \mathbf{u}$, consider the map $\Lambda' \rightarrow p(\Lambda')$. This is surjective, and $H_*(\Lambda' \cap I) \otimes \mathbb{Q}_l = 0$, since $(\Lambda' \cap I) \otimes \mathbb{Q}_l \cong I$. As $H_*(I) = 0$, we may choose a Δ -equivariant lattice $\Lambda' \cap I < M < I$ such that $H_*(M/lM) = 0$. Let $\Lambda := \Lambda' + M$, noting that this is an admissible lattice (p being small), with the maps $\Lambda/l^n \rightarrow p(\Lambda)/l^n$ all acyclic.

In order to show that $G(X)^{L,\rho,\text{Mal}}$ is cofibrant, take an arbitrary map $f : G(X)^{\hat{L}} \rightarrow \exp(\mathfrak{v}) \rtimes \Delta$ over ρ ; this must factor through $\exp(p(\Lambda'))$ for some admissible lattice $\Lambda' < \mathfrak{u}$, and we may replace Λ by Λ' as above. It therefore suffices to show that the corresponding map

$$f : X \rightarrow \bar{W}(\exp(p(\Lambda)) \rtimes \Delta)$$

in $\text{pro}(\mathbb{S})$ lifts to $\bar{W}(\exp(\Lambda) \rtimes \Delta)$. For each $n \in \mathbb{N}$, we have a map

$$f_n : X_{\alpha(n)} \rightarrow \bar{W}(\exp(p(\Lambda)/l^n) \rtimes \Delta),$$

and these are compatible with the structural morphisms.

We now prove existence of the lift by induction on n . Assume we have $g_n : X_{\alpha(n)} \rightarrow \bar{W}(\exp(\Lambda/l^n) \rtimes \Delta)$, such that $p \circ g_n = f_n$. This gives a map

$$(f_{n+1}, g_n) : X_{\alpha(n)} \rightarrow \bar{W}(\exp((p(\Lambda)/l^{n+1}) \times_{p(\Lambda)/l^n} (\Lambda/l^n)) \rtimes \Delta).$$

However, $\Lambda/l^{n+1} \rightarrow (p(\Lambda)/l^{n+1}) \times_{p(\Lambda)/l^n} (\Lambda/l^n)$ is an acyclic small extension, so

$$\bar{W}(\exp(\Lambda/l^{n+1}) \rtimes \Delta) \rightarrow \bar{W}(\exp((p(\Lambda)/l^{n+1}) \times_{p(\Lambda)/l^n} (\Lambda/l^n)) \rtimes \Delta)$$

is a trivial fibration, allowing us to construct a lift $g_{n+1} : X_{\alpha(n+1)} \rightarrow \bar{W}(\exp(\Lambda/l^{n+1}) \rtimes \Delta)$. This completes the proof that $G(X)^{L,\rho,\text{Mal}}$ is cofibrant.

Finally, if V is an R -representation then $H^{n+1}(G(X)^{L,\rho,\text{Mal}}, V)$ is the coequaliser of the diagram

$$\text{Hom}_{s\text{AGpd}R}(G(X)^{L,\rho,\text{Mal}}, (N^{-1}V[-n])^{\Delta^1}) \rightrightarrows \text{Hom}_{s\text{AGpd}R}(G(X)^{L,\rho,\text{Mal}}, N^{-1}V[-n]).$$

For a Δ -equivariant lattice $\Lambda \subset V$, this is the direct limit over m of

$$\text{Hom}_{\text{pro}(\mathbb{S}|\bar{W}R)}(X, \bar{W}((N^{-1}l^{-m}\Lambda[-n])^{\Delta^1} \rtimes R)) \rightrightarrows \text{Hom}_{\text{pro}(\mathbb{S}|\bar{W}R)}(X, \bar{W}(N^{-1}l^{-m}\Lambda[-n] \rtimes R)).$$

Hence

$$H^{n+1}(G(X)^{L,\rho,\text{Mal}}, V) \cong \varinjlim_m H^{n+1}(X, l^{-m}\Lambda) = H^{n+1}(X, \Lambda) \otimes \mathbb{Q}_l = H^{n+1}(X, V),$$

as required. \square

Definition 2.26. Given X and ρ as above, define the relative Malcev homotopy type

$$X^{\rho,\text{Mal}} := G(X)^{P,\rho,\text{Mal}},$$

where P is the set of all primes, noting that this is isomorphic to $G(X)^{L,\rho,\text{Mal}}$, by Proposition 2.25.

Define

$$X^{L,\text{alg}} := G(X)^{L,\text{alg}}.$$

Remark 2.27. Note that if $X \in \mathbb{S}$, this definition of Malcev completion differs slightly from the Malcev homotopy type $X^{\rho,\text{Mal}}$ of [Pri3] Definition 3.16, which is given by $G(X)^{\rho,\text{Mal}}$. However, the following lemma rectifies the situation.

Lemma 2.28. *For $X \in \mathbb{S}$ and $\rho : \pi_f(X)^{\hat{L}} \rightarrow R(\mathbb{Q}_l)$ Zariski-dense and continuous, there is a canonical map*

$$G(X)^{\rho,\text{Mal}} \rightarrow G(X)^{L,\rho,\text{Mal}};$$

this is a quasi-isomorphism whenever the groups $H^n(X, V)$ are finite-dimensional for all finite-dimensional R -representations V .

Proof. Existence of the map is immediate. To see that it gives a quasi-isomorphism, we need only look at cohomology groups. Given an R -representation V corresponding to a local system \mathbb{V} over \mathbb{Q}_l on X , the map on cohomology groups is

$$H^*(X^{\hat{L}}, \mathbb{V}) \rightarrow H^*(X, \mathbb{V});$$

this is an isomorphism by Remark 2.23. \square

Definition 2.29. Define pro-algebraic and relative homotopy groups by $\varpi_n(X^{\hat{L}}) := \pi_{n-1}(G(X)^{L, \text{alg}})$ and $\varpi_n(X^{\rho, \text{Mal}}) := \pi_{n-1}(G(X)^{P, \rho, \text{Mal}})$.

Define pro-algebraic and relative fundamental groupoids by $\varpi_f(X^{\hat{L}}) := \pi_f(X)^{L, \text{alg}}$ and $\varpi_f(X^{\rho, \text{Mal}}) := \widehat{\pi_f X}^{\rho, \text{Mal}}$.

Define $\varpi_f(\hat{X}), \varpi_n(\hat{X})$ by the convention that $\hat{X} = X^{\hat{P}}$, for P the set of all primes.

Corollary 2.30. *A map $f : X \rightarrow Y$ in $\text{pro}(\mathbb{S})$, with X_0, Y_0 discrete, induces an isomorphism*

$$f^{L, \text{alg}} : X^{L, \text{alg}} \rightarrow Y^{L, \text{alg}}$$

of homotopy types if and only if the following conditions hold:

- (1) f^* induces an equivalence between the categories of finite-dimensional semisimple continuous \mathbb{Q}_l -representations of $(\pi_f X)^{\hat{L}}$ and $(\pi_f Y)^{\hat{L}}$;
- (2) for all finite-dimensional semisimple continuous \mathbb{Q}_l -representations V , of $\pi_f Y$ the maps

$$f^* : H^*(Y, V) \rightarrow H^*(X, f^*V)$$

are isomorphisms.

2.4. Equivariant cochains.

Definition 2.31. Let $\mathcal{E}(R)$ be the full subcategory of AGpd consisting of unipotent extensions of R .

In [Pri3] §3, a contravariant equivalence is established between the homotopy category $\text{Ho}(s\mathcal{E}(R)) \subset \text{Ho}(\text{AGpd} \downarrow R)$ of simplicial unipotent extensions of R , and the homotopy category $\text{Ho}(c\text{Alg}(R)_0)$ of R -representations in connected cosimplicial algebras. In [Pri3] Theorem 3.55, it was shown that the homotopy type of $G(X)^{\rho, \text{Mal}}$, for $X \in \mathbb{S}$, corresponds to the algebra

$$C^\bullet(X, \mathbb{O}(R))$$

of equivariant cochains, where $\mathbb{O}(R)$ is the local system on X corresponding to the left action of $\pi_f X$ on the structure ring $O(R)$ of R .

Definition 2.32. Given a pro-simplicial set X , and a map $\pi_f X \rightarrow \Gamma$ to a pro-groupoid with discrete objects, define the covering system \tilde{X} by

$$\tilde{X}(a) := X \times_{B\Gamma} B(\Gamma \downarrow a) \in \text{pro}(\mathbb{S})$$

for $a \in \text{Ob } \Gamma$, noting that this is equipped with a natural associative action $\Gamma(a, b) \times \tilde{X}(a) \rightarrow \tilde{X}(b)$ in $\text{pro}(\mathbb{S})$.

Definition 2.33. Given $\pi_f X \rightarrow \Gamma$ as above, with a continuous representation S of Γ in pro-sets (i.e. $S(a) \in \text{pro}(\text{Set})$ for $a \in \text{Ob } \Gamma$, equipped with an associative action $\Gamma(a, b) \times S(a) \rightarrow S(b)$ of pro-sets), define the cosimplicial set $C^\bullet(X, S)$ by

$$C^n(X, S) := \text{Hom}_{\Gamma, \text{pro}(\text{Set})}(\tilde{X}_n, S).$$

Lemma 2.34. *If Λ is a Γ -representation in pro-simplicial groups such that $\Lambda \rtimes \Gamma \in \text{pro}(s\text{Gpd})$, then*

$$\text{Hom}_{\Gamma, \text{pro}(\mathbb{S})}(\tilde{X}, \bar{W}\Lambda) \cong \text{Hom}_{\text{pro}(\mathbb{S}) \downarrow B\Gamma}(X, \bar{W}(\Lambda \rtimes \Gamma)).$$

Proof. The calculation is the same as for [Pri3] Lemma 3.53. \square

Definition 2.35. Given a pro-algebraic groupoid G over \mathbb{Z}_l , define $O(G)$ to be the $G \times G$ -representation given by global sections of the structure sheaf of G , equipped with its left and right G -actions.

Given a representation $\rho : \pi_f X \rightarrow G(\mathbb{Z}_l)$, let $\mathbb{O}(G)$ be the G -representation in \mathbb{Z}_l -local systems on X given by pulling $O(G)$ back along its right G -action.

Definition 2.36. Given X, L, ρ, R as in Proposition 2.25, let $R_{\mathbb{Z}_l}$ be the \mathbb{Z}_l -model for R constructed in Proposition 2.14, and set

$$\mathbf{C}^\bullet(X, \mathbb{O}(R)) := \mathbf{C}^\bullet(X, \mathbb{O}(R_{\mathbb{Z}_l})) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.$$

Theorem 2.37. *For X, L, ρ, R as in Proposition 2.25, the relative Malcev homotopy type*

$$G(X)^{L, \rho, \text{Mal}} \in s\text{AGpd} \downarrow R$$

corresponds under the equivalence of [Pri3] §3 to the R -representation

$$\mathbf{C}^\bullet(X, \mathbb{O}(R))$$

in cosimplicial k -algebras.

Proof. We need to show that, for $\mathbf{u} \in s\mathcal{N}(R)$,

$$\text{Hom}_{s\text{AGpd} \downarrow R}(G(X)^{L, \rho, \text{Mal}}, \exp(\mathbf{u}) \rtimes R) \cong \text{Hom}_{s\text{Aff}(R)}(\text{Spec } \mathbf{C}^\bullet(X, \mathbb{O}(R)), \bar{W}(\exp(\mathbf{u}))).$$

Adapting the proof of Proposition 2.25, we know that

$$\text{Hom}_{s\text{AGpd} \downarrow R}(G(X)^{L, \rho, \text{Mal}}, \exp(\mathbf{u}) \rtimes R) \cong \varinjlim_{\Lambda} \text{Hom}_{\text{pro}(\mathbb{S})}(X, \bar{W}(\exp(\Lambda) \rtimes R_{\mathbb{Z}_l}(\mathbb{Z}_l)))_{BR_{\mathbb{Z}_l}(\mathbb{Z}_l)},$$

where the limit is taken over $\Lambda \subset \mathbf{u}$ admissible. By Lemma 2.34,

$$\text{Hom}_{\text{pro}(\mathbb{S})}(X, \bar{W}(\exp(\Lambda) \rtimes R_{\mathbb{Z}_l}(\mathbb{Z}_l)))_{BR_{\mathbb{Z}_l}(\mathbb{Z}_l)} \cong \text{Hom}_{R_{\mathbb{Z}_l}(\mathbb{Z}_l), \text{pro}(\mathbb{S})}(\tilde{X}, \bar{W} \exp(\Lambda)).$$

If we regard $\exp(\Lambda)$ as the \mathbb{Z}_l -valued points of the group scheme $\exp(\Lambda)(A) := \exp(\Lambda \otimes A)$, then this is an affine space, so

$$\text{Hom}_{\text{pro}(\mathbb{S})}(\tilde{X}, \bar{W} \exp(\Lambda)) \cong \text{Hom}_{s\text{Aff}_{\mathbb{Z}_l}}(\text{Spec } \text{Hom}_{\text{pro}(\text{Set})}(\tilde{X}, \mathbb{Z}_l), \bar{W} \exp(\Lambda)).$$

Since $\Lambda \cong \Lambda \otimes^R O(R_{\mathbb{Z}_l})$, we then have

$$\text{Hom}_{R_{\mathbb{Z}_l}(\mathbb{Z}_l), \text{pro}(\mathbb{S})}(\tilde{X}, \bar{W} \exp(\Lambda)) \cong \text{Hom}_{s\text{Aff}(R_{\mathbb{Z}_l})}(\text{Spec } \mathbf{C}^\bullet(X, \mathbb{O}(R_{\mathbb{Z}_l})), \bar{W} \exp(\Lambda)).$$

The map

$$\begin{aligned} & \varinjlim_{\Lambda} \text{Hom}_{s\text{Aff}(R_{\mathbb{Z}_l})}(\text{Spec } \mathbf{C}^\bullet(X, \mathbb{O}(R_{\mathbb{Z}_l})), \bar{W} \exp(\Lambda)) \\ & \rightarrow \varinjlim_{\Lambda} \text{Hom}_{s\text{Aff}(R_{\mathbb{Z}_l})}(\text{Spec } \mathbf{C}^\bullet(X, \mathbb{O}(R_{\mathbb{Z}_l})) \otimes \mathbb{Q}_l, \bar{W} \exp(\Lambda)) \end{aligned}$$

is clearly injective. However, since there exists a lattice Λ' with $l^{-n}\Lambda \subset \Lambda'$, the map must also be surjective. Finally, note that

$$\begin{aligned} & \text{Hom}_{s\text{Aff}(R_{\mathbb{Z}_l})}(\text{Spec } \mathbf{C}^\bullet(X, \mathbb{O}(R_{\mathbb{Z}_l})) \otimes \mathbb{Q}_l, \bar{W} \exp(\Lambda)) \\ & = \text{Hom}_{s\text{Aff}(R)}(\text{Spec } \mathbf{C}^\bullet(X, \mathbb{O}(R)), \bar{W} \exp(\Lambda \otimes \mathbb{Q}_l)), \end{aligned}$$

as required. \square

Remarks 2.38. Note that if we take a scheme X , then as in [Pri3] Remark 3.55, $C^\bullet(X_{\text{ét}}, \mathbb{V})$ is a Godement resolution for the cohomology of \mathbb{V} . Under the comparison of [Pri3] Corollary 3.57, this shows that for an algebraic variety X , $\widehat{G(X_{\text{ét}})}^{\text{alg}}$ agrees with the l -adic homotopy type discussed in [Toë] §3.5.3.

Given any morphism $\rho : \varpi_f(\widehat{X_{\text{ét}}})^{\text{red}} \rightarrow R$ to a reductive group, there is a forgetful functor $\rho^\# : s\hat{\mathcal{N}}(R) \rightarrow s\hat{\mathcal{N}}(\varpi_f(\widehat{X_{\text{ét}}})^{\text{red}})$. If we write $\mathbb{L}\rho_\#$ for the derived left adjoint and ρ is surjective, then $R_u(\widehat{G(X_{\text{ét}})})^{\rho, \text{Mal}} = \mathbb{L}\rho_\# R_u(\widehat{G(X_{\text{ét}})})^{\text{alg}}$. Note that for \mathcal{C} a Tannakian subcategory of $\text{FDRep}(\varpi_f(\widehat{X_{\text{ét}}})^{\text{red}})$, with corresponding groupoid G , the homotopy type $X_{\mathcal{C}_{\text{ét}}}$ of [Ols] 1.5 is equivalent to $\mathbb{L}\rho_\# R_u(\widehat{G(X)})^{\text{alg}}$, for $\rho : \varpi_f(\widehat{X_{\text{ét}}})^{\text{red}} \rightarrow G$.

2.5. Comparison with Artin-Mazur homotopy groups.

Definition 2.39. Recall that a discrete groupoid Γ is said to be good (in the sense of [Pri3] Definition 3.18) with respect to a Zariski-dense representation $\rho : \Gamma \rightarrow R(k)$ to a reductive pro-algebraic groupoid if the map

$$H^n(\Gamma^{\rho, \text{Mal}}, V) \rightarrow H^n(\Gamma, V)$$

is an isomorphism for all n and all finite-dimensional $\Gamma^{\rho, \text{Mal}}$ -representations V .

Definition 2.40. Say that a locally profinite groupoid Γ is good with respect to a continuous Zariski-dense representation $\rho : \Gamma \rightarrow R(\mathbb{Q}_l)$ to a reductive pro-algebraic groupoid if the map

$$H^n(\Gamma^{\rho, \text{Mal}}, V) \rightarrow H^n(\Gamma, V)$$

is an isomorphism for all n and all finite-dimensional $\Gamma^{\rho, \text{Mal}}$ -representations V .

If Γ is good relative to Γ^{red} , then we say that Γ is algebraically good.

Lemma 2.41. *If Γ is a finitely presented L -good discrete groupoid and $\rho : \Gamma^{\hat{L}} \rightarrow R(\mathbb{Q}_l)$ as above, with $l \in L$ and Γ good relative to $\rho : \Gamma \rightarrow R(\mathbb{Q}_l)$ in the sense of [Pri3] Definition 3.18, then $\Gamma^{\hat{L}}$ is good relative to ρ .*

Proof. Take a finite-dimensional R -representation V . By Lemma 2.28, $(B\Gamma)^{\rho, \text{Mal}} \simeq (B\Gamma)^{L, \rho, \text{Mal}}$. Since Γ is relatively good, we have $(B\Gamma)^{\rho, \text{Mal}} \simeq \Gamma^{\rho, \text{Mal}}$, so $(B\Gamma)^{L, \rho, \text{Mal}} \simeq \Gamma^{\rho, \text{Mal}}$. Moreover, as Γ is L -good, $(B\Gamma)^{L, \rho, \text{Mal}} = (B\Gamma^{\hat{L}})^{\rho, \text{Mal}}$.

We have therefore shown that $(B\Gamma^{\hat{L}})^{\rho, \text{Mal}} \simeq \pi_f(B\Gamma^{\hat{L}})^{\rho, \text{Mal}} = \Gamma^{L, \rho, \text{Mal}}$, so we have isomorphisms

$$H^*(\Gamma^{L, \rho, \text{Mal}}, V) \rightarrow H^*(\Gamma^{\hat{L}}, V),$$

as required. \square

Lemma 2.42. *Assume that for all $x \in \text{Ob } \Gamma$, $\Gamma(x, x)$ is finitely presented as a profinite group, with $H^n(\Gamma, -)$ commuting with filtered direct limits of $\Gamma^{\rho, \text{Mal}}$ -representations, and $H^n(\Gamma, V)$ finite-dimensional for all finite-dimensional $\Gamma^{\rho, \text{Mal}}$ -representations V .*

Then Γ is good with respect to ρ if and only if for any finite-dimensional $\Gamma^{\rho, \text{Mal}}$ -representation V , and $\alpha \in H^n(\Gamma, V)$, there exists an injection $f : V \rightarrow W_\alpha$ of finite-dimensional $\Gamma^{\rho, \text{Mal}}$ -representations, with $f(\alpha) = 0 \in H^n(\Gamma, W_\alpha)$.

Proof. The proof of [KPT] Lemma 4.15 carries over to this context. \square

Examples 2.43. A profinite group Γ is good with respect to a representation $\rho : \Gamma^{\hat{L}} \rightarrow R$ whenever any of the following holds:

- (1) Γ is finite, or $\Gamma^{\hat{L}} \cong \Delta^{\hat{L}}$, for Δ a finitely generated free discrete group.
- (2) $\Gamma^{\hat{L}} \cong \Delta^{\hat{L}}$, for Δ a finitely generated nilpotent discrete group.
- (3) $\Gamma^{\hat{L}} \cong \Delta^{\hat{L}}$, for Δ the fundamental group of a compact Riemann surface. In particular, this applies if Γ is the fundamental group of a smooth projective curve C/k , for k a separably closed field whose characteristic is not in L .
- (4) If $1 \rightarrow F \rightarrow \Gamma \rightarrow \Pi \rightarrow 1$ is an exact sequence of groups, with F finite, assume that $\Pi^{\hat{L}}$ is good relative to $R/\overline{\rho(F)}$, where $\overline{}$ denotes Zariski closure. Then Γ is good relative to ρ .

Proof. Combine Lemma 2.41 with Examples 1.12 and [Pri3] Examples 3.20. \square

Theorem 2.44. *If L is a set of primes containing l and $X \in \hat{\mathcal{H}}$ has fundamental groupoid $\pi_f X = \Gamma$, equipped with a continuous Zariski-dense representation $\rho : \Gamma^{\hat{L}} \rightarrow R(\mathbb{Q}_l)$ to a reductive pro-algebraic groupoid for which:*

- (1) $\Gamma^{\hat{L}}$ is good with respect to ρ ,
- (2) $\pi_n(X^{\hat{L}}, -) \otimes_{\hat{\mathbb{Z}}} \mathbb{Q}_l$ is finite-dimensional for all $n > 1$, and
- (3) the $\Gamma^{\hat{L}}$ -representation $\pi_n(X^{\hat{L}}, -) \otimes_{\hat{\mathbb{Z}}} \mathbb{Q}_l$ is an extension of R -representations (i.e. a $\Gamma^{L,\rho,\text{Mal}}$ -representation),

then the canonical map

$$\pi_n(X^{\hat{L}}, -) \otimes_{\hat{\mathbb{Z}}} \mathbb{Q}_l \rightarrow \varpi_n(X^{L,\rho,\text{Mal}})$$

is an isomorphism for all $n > 1$.

Proof. This is essentially the same as [Pri3] Theorem 1.58. The only difference in the proof is that we cannot immediately appeal to the Curtis convergence theorem to show that for any pro-discrete abelian group π and $n \geq 2$, the map

$$G(K(\pi, n))^{L,\text{alg}} \rightarrow N^{-1}(\hat{\pi} \otimes_{\hat{\mathbb{Z}}} \mathbb{Q}_l[1-n])$$

is a weak equivalence of simplicial unipotent groups.

Instead, observe that we may replace π by $\pi^{\hat{l}}$, so assume that π is a pro- l group. Since $\pi \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ is finite-dimensional, we may write $\pi = \nu^{\hat{l}}$, for ν an abelian group of finite rank. On cohomology, we have maps

$$H^*(N^{-1}(\pi \otimes_{\mathbb{Z}_l} \mathbb{Q}_l[1-n]), \mathbb{Q}_l) \rightarrow H^*(K(\pi, n), \mathbb{Q}_l) \rightarrow H^*(K(\nu, n), \mathbb{Q}_l). \quad (\dagger)$$

By [Qui2] Theorem I.3.4, the Lie algebra $\nu \otimes_{\mathbb{Z}} \mathbb{Q}_l[1-n]$ is the \mathbb{Q}_l homotopy type of $K(\nu, n)$. Since $\pi \otimes_{\mathbb{Z}_l} \mathbb{Q}_l = \nu \otimes_{\mathbb{Z}} \mathbb{Q}_l$, the composite is an isomorphism in (\dagger) , while the second map is an isomorphism by Lemma 1.18. Thus the first map is an isomorphism, as required. \square

2.6. Comparison of homotopy types for complex varieties. Let X_{\bullet} be a simplicial scheme of finite type over \mathbb{C} . To this we may associate the étale homotopy type $X_{\text{ét}} \in \text{pro}(\mathbb{S})$ (as in Example 2.24). There is also an analytic homotopy type $X_{\text{an}} := \text{diag Sing}(X_{\bullet}(\mathbb{C})) \in \mathbb{S}$, for $\text{Sing} : \text{Top} \rightarrow \mathbb{S}$ the functor

$$\text{Sing}(Y)_n := \text{Hom}_{\text{Top}}(|\Delta^n|, Y)$$

of homomorphisms from geometric simplices, and diag the diagonal functor on bisimplicial sets.

Lemma 2.45. *If G is a pro-algebraic group over \mathbb{Q}_l , and $\rho : \pi_f(X_{\text{an}}) \rightarrow G(\mathbb{Q}_l)$ a representation with compact image (for the l -adic topology on $G(\mathbb{Q}_l)$), then ρ factorises canonically through $\widehat{\pi_f(X_{\text{ét}})}$, giving a continuous representation*

$$\rho : \widehat{\pi_f(X_{\text{ét}})} \rightarrow G(\mathbb{Q}_l).$$

Proof. It follows from [Fri] Theorem 8.4 that

$$\widehat{\pi_f(X_{\text{ét}})} \cong \widehat{\pi_f(X_{\text{an}})}.$$

Since $G(\mathbb{Q}_l)$ is totally disconnected, any compact subgroup is profinite, completing the proof. \square

Now, given a reductive pro-algebraic groupoid R , and $\rho : \pi_f(X_{\mathbb{C}}) \rightarrow R(\mathbb{Q}_l)$ with compact Zariski-dense image, we may compare the relative Malcev homotopy type $X_{\text{an}}^{\rho, \text{Mal}}$ of [Pri3] Definition 3.16 with the relative Malcev homotopy type $X_{\text{ét}}^{\rho, \text{Mal}}$ of Definition 2.26, since both are objects of $\text{Ho}(s\mathcal{E}(R))$.

Theorem 2.46. *For X, ρ as above, there is a canonical isomorphism*

$$X_{\text{an}}^{\rho, \text{Mal}} \cong X_{\text{ét}}^{\rho, \text{Mal}}.$$

Proof. We adapt [Fri] Theorem 8.4, which constructs a new homotopy type $X_{s, \text{ét}}$, and gives morphisms

$$X_{\text{ét}} \leftarrow X_{s, \text{ét}} \rightarrow X_{\text{an}}$$

in $\text{pro}(\mathbb{S})$, inducing weak equivalences on profinite completions. By Lemma 2.28, $X_{\text{an}}^{\rho, \text{Mal}}$ is quasi-isomorphic to $\widehat{G(X_{\text{an}})}^{\rho, \text{Mal}}$. By Corollary 1.16, the maps $\widehat{G(X_{\text{ét}})}^{\rho, \text{Mal}} \leftarrow \widehat{G(X_{s, \text{ét}})}^{\rho, \text{Mal}} \rightarrow \widehat{G(X_{\text{an}})}^{\rho, \text{Mal}}$ are then quasi-isomorphisms. \square

Remarks 2.47. In particular, this shows that there is an action of the Galois group $\text{Gal}(\mathbb{C}/K)$ on the relative Malcev homotopy groups $\varpi_n(X_{\text{an}}^{\rho, \text{Mal}})$ whenever X is defined over a number field K . The question of when this action is continuous will be addressed in §4.

It seems possible that the conditions of [Pri3] Theorem 3.21 might be verified in some cases where those of Theorem 2.44 do not hold, giving $\varpi_n(X_{\text{an}}^{\rho, \text{Mal}}) \cong \pi_n(X_{\text{an}}) \otimes_{\mathbb{Z}} \mathbb{Q}_l$, but no such examples are known to the author.

3. RELATIVE AND FILTERED HOMOTOPY TYPES

3.1. Outer actions on pro-algebraic homotopy types. Fix a \mathbb{Q}_l -algebra A , and a reductive pro-algebraic groupoid R over \mathbb{Q}_l .

Definition 3.1. Define $c\text{Alg}_A(R)$ (resp. $DG\text{Alg}_A(R)$) to be the comma category $A \downarrow c\text{Alg}(R)$ (resp. $A \downarrow DG\text{Alg}(R)$), with its standard model structure. Denote the opposite category by $s\text{Aff}_A(R)$ (resp. $dg\text{Aff}_A(R)$).

The following is standard (see e.g. [Pri3] Proposition 3.2).

Proposition 3.2. *There is a denormalisation functor $D : DG\text{Alg}_A(R) \rightarrow c\text{Alg}_A(R)$, which is a right Quillen equivalence, giving the following equivalence of categories:*

$$\text{Ho}(dg\text{Aff}_A(R))_0 \xrightleftharpoons[\mathbb{R}(\text{Spec } D^*)]{\text{Spec } D} \text{Ho}(s\text{Aff}_A(R))_0.$$

3.1.1. Lie algebras.

Definition 3.3. Recall that a Lie coalgebra C is said to be conilpotent if the iterated cobracket $\Delta_n : C \rightarrow C^{\otimes n}$ is 0 for sufficiently large n . A Lie coalgebra C is ind-conilpotent if it is a filtered direct limit (or, equivalently, a nested union) of conilpotent Lie coalgebras.

Definition 3.4. Recall from [Pri3] Definition 5.8 that $\hat{\mathcal{N}}_A(R)$ is defined to be opposite to the category of R -representations in ind-conilpotent Lie coalgebras over A .

Similarly, $dg\hat{\mathcal{N}}_A(R)$ is opposite to the category of R -representations in ind-conilpotent \mathbb{N}_0 -graded cochain Lie coalgebras over A , and $s\hat{\mathcal{N}}_A(R)$ consists of simplicial objects in $\hat{\mathcal{N}}_A(R)$.

Note that $\hat{\mathcal{N}}_k(R) \cong \hat{\mathcal{N}}(R)$, and that there is a continuous functor $\hat{\mathcal{N}}(R) \rightarrow \hat{\mathcal{N}}_A(R)$ given by $C^\vee \mapsto (C \otimes_k A)^\vee$. We denote this by $\mathfrak{g} \mapsto \mathfrak{g} \hat{\otimes} A$.

Remark 3.5. Observe that $\mathfrak{g} \in \hat{\mathcal{N}}_A(R)$ can be regarded as an object of the category $\text{Aff}_A(R) := \text{Aff}(R) \downarrow \text{Spec } A$ or R -representations in affine A -schemes, by regarding it as the functor

$$\mathfrak{g}(B) := \text{Hom}_{A,R}(\mathfrak{g}^{\text{opp}}, B),$$

for $B \in \text{Alg}_A(R) := A \downarrow \text{Alg}(R)$. In fact, $\mathfrak{g}(B)$ is then a Lie algebra over B , so the Campbell-Baker-Hausdorff formula defines a group structure on $\mathfrak{g}(B)$, and the resulting group is denoted by $\exp(\mathfrak{g})(B)$. Thus $\exp(\mathfrak{g})$ is an R -representation in affine group schemes over A (i.e. a group object of $\text{Aff}_A(R)$).

The following is [Pri3] Lemma 5.9:

Lemma 3.6. *There is a closed model structure on $dg\hat{\mathcal{N}}_A(R)$ in which a morphism $f : \mathfrak{g} \rightarrow \mathfrak{h}$ is a fibration or a weak equivalence whenever the underlying map $f^\vee : \mathfrak{h}^\vee \rightarrow \mathfrak{g}^\vee$ in $DG\text{Mod}_A(R)$ is a cofibration or a weak equivalence.*

Remark 3.7. It follows from the construction in [Pri3] Lemma 5.9 that for cofibrant objects $\mathfrak{g} \in dg\hat{\mathcal{N}}(R)$ (taking A to be a field), \mathfrak{g}^\vee is freely cogenerated as a graded Lie coalgebra. Thus $\mathfrak{g}^\vee[-1]$ is a positively graded strong homotopy commutative algebra without unit (in the sense of [Kon] Lectures 8 and 15), and a choice of cogenerators on \mathfrak{g}^\vee is the same as a positively graded E_∞ algebra — this is an aspect of Koszul duality.

By [Pri3] Proposition 4.12, there is a normalisation functor $N : s\hat{\mathcal{N}}(R) \rightarrow dg\hat{\mathcal{N}}(R)$, which is a right Quillen equivalence, so gives equivalences on homotopy categories. In [Pri3] Corollary 4.41, an equivalence was given between the full subcategory $\text{Ho}(dg\text{Aff}(R))_0$ of the homotopy category $\text{Ho}(dg\text{Aff}(R))$, consisting of objects $\text{Spec } B$ with $B^0 = \mathbb{Q}_l$, and the category $dg\mathcal{M}(R)$ defined to have the objects of $dg\hat{\mathcal{N}}(R)$, with morphisms given by

$$\text{Hom}_{dg\mathcal{M}(R)}(\mathfrak{g}, \mathfrak{h}) = \text{Hom}_{\text{Ho}(dg\hat{\mathcal{N}}(R))}(\mathfrak{g}, \mathfrak{h}) / \exp(\mathfrak{h}_0^R),$$

where \mathfrak{h}_0^R is the Lie algebra $\text{Hom}_R(\mathfrak{h}_0^\vee, \mathbb{Q}_l)$, acting by conjugation on the set of homomorphisms. The equivalences are denoted by $\bar{W} : dg\mathcal{M}(R) \rightarrow \text{Ho}(dg\text{Aff}(R))_0$, and $\bar{G} : \text{Ho}(dg\text{Aff}(R))_0 \rightarrow dg\mathcal{M}(R)$.

Although we do not have a precise analogue of this result for $\text{Ho}(dg\text{Aff}_A(R))$, we have the following:

Lemma 3.8. *Given $X \in \mathrm{Ho}(dg\mathrm{Aff}(R))_0$ and $\mathfrak{g} \in dg\hat{\mathcal{N}}(R)$,*

$$\mathrm{Hom}_{\mathrm{Ho}(dg\mathrm{Aff}_A(R))}(X \otimes A, \bar{W}\mathfrak{g} \otimes A) \cong \mathrm{Hom}_{\mathrm{Ho}(dg\hat{\mathcal{N}}_A(R))}(\bar{G}(X) \hat{\otimes} A, \mathfrak{g} \hat{\otimes} A) / \exp(\mathfrak{g}_0^R \hat{\otimes} A).$$

Proof. The proof of [Pri3] Proposition 3.48 adapts to this context. \square

3.2. Outer actions.

Definition 3.9. Given $G \in s\mathcal{E}(R)$, recall from [Pri3] Definition 5.12 that there is a group presheaf $\mathrm{ROut}(G)$ over \mathbb{Q}_l , with

$$\mathrm{ROut}(G)(\mathbb{Q}_l) \cong \mathrm{Aut}_{\mathrm{Ho}(s\mathcal{E}(R))}(\mathbb{Q}_l).$$

If $G \in s\mathcal{E}(R)$ is such that $H^i(G, V)$ is finite-dimensional for all i and all finite-dimensional irreducible R -representations V , then by [Pri3] Theorem 5.13, $\mathrm{ROut}(G)$ is a pro-algebraic group over \mathbb{Q}_l . For $G \in s\mathrm{AGpd}$, we define $\mathrm{ROut}(G)$ by taking $R = G^{\mathrm{red}}$.

Definition 3.10. Given a pro-algebraic groupoid G , we may extend the automorphism group $\mathrm{Aut}(G)$ to a group presheaf over \mathbb{Q}_l , by setting

$$\mathrm{Aut}(G)(A) := \mathrm{Aut}_A(G \times_{\mathrm{Spec} \mathbb{Q}_l} \mathrm{Spec} A).$$

Lemma 3.11. *For $G \in s\mathcal{E}(R)$, the group $\mathrm{Out}(G)$ can be canonically extended to a group presheaf over \mathbb{Q}_l , also denoted $\mathrm{Out}(G)$ (with $\mathrm{Out}(G)(\mathbb{Q}_l) = \mathrm{Out}(G)$), such that the exact sequence above extends to an exact sequence*

$$1 \rightarrow \mathrm{ROut}(G) \rightarrow \mathrm{Out}(G) \xrightarrow{\alpha} \mathrm{Aut}(G^{\mathrm{red}}) \rightarrow 1,$$

where $\mathrm{Aut}(G^{\mathrm{red}})$ is given the algebraic structure of Definition 3.10.

If $H^i(G, V)$ is finite-dimensional for all i and all finite-dimensional irreducible R -representations V , then α is fibred in affine schemes.

Proof. Let $R = G^{\mathrm{red}}$, take $Y \in \mathrm{Ho}(dg\mathrm{Aff}(R))$ corresponding to G under the equivalence of [Pri3] Theorem 4.41 and define

$$\mathrm{Out}(G)(A) := \{(f, \theta) : f \in \mathrm{Aut}(R)(A), \theta \in \mathrm{Iso}_{\mathrm{Ho}(dg\mathrm{Aff}_A(R))}(Y \otimes A, f^\# Y \otimes A)\}.$$

We may now take a minimal model \mathfrak{m} for $\bar{G}(Y) \in dg\hat{\mathcal{N}}(R)$, and observe that Lemma 3.8 then gives

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Ho}(dg\mathrm{Aff}_A(R))}(Y \otimes A, f^\# Y \otimes A) &\cong \mathrm{Hom}_{\mathrm{Ho}(dg\mathrm{Aff}_A(R))}(Y \otimes A, f^\# \bar{W}\mathfrak{m} \otimes A) \\ &\cong \mathrm{Hom}_{\mathrm{Ho}(dg\hat{\mathcal{N}}_A(R))}(\bar{G}(Y) \hat{\otimes} A, \mathfrak{m} \hat{\otimes} A) / \exp(\mathfrak{m}_0^R \hat{\otimes} A) \\ &\cong \mathrm{Hom}_{\mathrm{Ho}(dg\hat{\mathcal{N}}_A(R))}(\mathfrak{m} \hat{\otimes} A, \mathfrak{m} \hat{\otimes} A) / \exp(\mathfrak{m}_0^R \hat{\otimes} A). \end{aligned}$$

The proof that α is fibred in affine schemes is now essentially the same as [Pri3] Theorem 5.13 (which deals with the fibre over $1 \in \mathrm{Aut}(R)$). \square

Definition 3.12. Given a pro-discrete group Γ , we say that a morphism $\Gamma \rightarrow \mathrm{Out}(G)$ is algebraic if it factors through a morphism $\Gamma^{\mathrm{alg}} \rightarrow \mathrm{Out}(G)$ of presheaves of groups.

Corollary 3.13. *If $H^i(G, V)$ is finite-dimensional for all i and all finite-dimensional irreducible R -representations V , with $\Gamma \rightarrow \mathrm{Aut}(G^{\mathrm{red}})$ algebraic, then $\Gamma \rightarrow \mathrm{Out}(G)$ is algebraic.*

Proof. We have $\Gamma^{\mathrm{alg}} \rightarrow \mathrm{Aut}(G^{\mathrm{red}})$, so $\theta : \Gamma \rightarrow (\Gamma^{\mathrm{alg}} \times_{\mathrm{Aut}(G^{\mathrm{red}})} \mathrm{Out}(G))(\mathbb{Q}_l)$. Since $\mathrm{Out}(G) \rightarrow \mathrm{Aut}(G^{\mathrm{red}})$ is fibred in affine schemes, the group on the right is pro-algebraic, so θ factors through Γ^{alg} , as required. \square

If $R = G^{\text{red}}$, observe that there is canonical action of $\text{Out}(G)$ on $\bigoplus_{x \in \text{Ob } R} H^*(G, O(R)(x))$. In fact, we have a homomorphism

$$\beta : \text{Out}(G) \rightarrow \text{Aut}(R) \times \text{Aut}\left(\bigoplus_{x \in \text{Ob } R} H^*(G, O(R)(x))\right)$$

of presheaves of groups.

Lemma 3.14. *If $H^i(G, V)$ is finite-dimensional for all i and all finite-dimensional irreducible R -representations V , then the kernel of β is a pro-unipotent pro-algebraic group.*

Proof. The kernel of β is just the kernel of

$$\text{ROut}(G) \rightarrow \text{Aut}_R(H^*(G, O(R))),$$

which is pro-unipotent by [Pri3] Theorem 5.13. \square

3.3. Filtered homotopy types.

3.3.1. Commutative algebras.

Definition 3.15. Given a \mathbb{Q}_l -algebra A and a reductive pro-algebraic groupoid R over \mathbb{Q}_l , define $FDG\text{Alg}_A(R)$ (resp. $Fc\text{Alg}_A(R)$) to consist of R -representations in non-negatively graded cochain (resp. cosimplicial) algebras B over A , equipped with an increasing exhaustive filtration $J_0 B \subset J_1 B \subset \dots$, of B as a DG (resp. cosimplicial) (R, A) -module, with the property that $(J_m B) \cdot (J_n B) \subset J_{m+n} B$. Morphisms are required to respect the filtration, and we assume that $1 \in J_0 B$.

Definition 3.16. Given $(B, J) \in FDG\text{Alg}_A(R)$ or $Fc\text{Alg}_A(R)$, there is a spectral sequence ${}_J E_*^{*,*}(B)$ associated to the filtration J , with

$${}_J E_1^{a,b}(B) = H^{a+b}(\text{Gr}_{-a}^J B).$$

We regard ${}_J E_1^{*,*}(B)$ as an object of $FDG\text{Alg}_A(R)$, with

$$J_m({}_J E_1^{*,*}(B))^n = \bigoplus_{r \leq m} {}_J E_1^{-r, n+r}(B),$$

noting that $d(J_m(E_1)^n) \subset J_{m-1}(E_1)^{n+1}$.

Definition 3.17. Define a map $f : B \rightarrow C$ to be a fibration if the maps $J_n f : J_n B \rightarrow J_n C$ are all surjective. A map f is a weak equivalence if the maps ${}_J E_1^{*,*}(f) : {}_J E_1^{*,*}(B) \rightarrow {}_J E_1^{*,*}(C)$ are all isomorphisms.

Lemma 3.18. *There are cofibrantly generated model structures on the categories $Fc\text{Mod}_A(R)$ and $FDG\text{Mod}_A(R)$ of non-negatively exhaustively filtered R -representations in cosimplicial A -modules and \mathbb{N}_0 -graded cochain A -modules, in which fibrations are surjections, and weak equivalences are isomorphisms on ${}_J E_1^{*,*}(C) = H^*(\text{Gr}_*^J C)$.*

Proof. Let $S_{n,m}$ denote the cochain complex (resp. the cosimplicial complex) consisting of (resp. whose normalisation consists of) A concentrated in degree n , with $J_m S_{n,m} = S_{n,m}$, $J_{m-1} S_{n,m} = 0$. Let $D_{n,m}$ denote the cochain complex (resp. the cosimplicial complex) consisting of (resp. whose normalisation consists of) A concentrated in degrees $n, n-1$ with differential d^{n-1} the identity and $J_m D_{n,m} = D_{n,m}$, $J_{m-1} D_{n,m} = 0$. By convention, $D_{0,m} = 0$. Note that there are natural maps $S_{n,m} \rightarrow D_{n,m}$.

For a set $\{V\}$ of representatives of irreducible R -representations, define I to be the set of morphisms $S_{n,m} \otimes V \rightarrow D_{n,m} \otimes V$, for $n \geq 0$. Define J to be the set of morphisms $0 \rightarrow D_{n,m} \otimes V$, for $n \geq 0$. Then we have a cofibrantly generated model structure, with I the generating cofibrations and J the generating trivial cofibrations, by verifying the conditions of [Hov] Theorem 2.1.19. \square

Lemma 3.19. *In the category $FDGMod_A(R) = FDGMod_{\mathbb{Q}_l}(R)$, all objects V are cofibrant, as is the shifted complex $V[-1]$.*

Proof. Given $V \in FDGMod_{\mathbb{Q}_l}(R)$, it will suffice to show that J_0V is cofibrant, and all the maps $J_{m-1}V \rightarrow J_mV$ are cofibrations, and likewise for $V[-1]$, since $V = \varinjlim J_mV$. To do this, we will show that V is a transfinite composition of pushouts of the generating cofibrations, excluding $S_{0,m} \otimes V \rightarrow D_{0,m} \otimes V$.

Now, since all R -representations are semisimple, we may decompose the complex $gr_m^J V$ as $gr_m^J V^n M^n \oplus N^n \oplus dN^{n-1}$, with $dM^* = 0$. By semisimplicity, we may also lift the R -modules M^i, N^i to $\tilde{M}^i, \tilde{N}^i \subset J_m V$. Now $d\tilde{M} \subset J_{m-1}V$, so the map $J_{m-1}V \rightarrow J_mV$ is a pushout of $\bigoplus_n (S_{n+1,m} \otimes \tilde{M}^n) \rightarrow \bigoplus_n (D_{n+1,m} \otimes \tilde{M}^n) \oplus \bigoplus_n (D_{n+1,m} \otimes \tilde{N}^n)$, and hence a cofibration. Since this argument also applies to $0 \rightarrow J_0V$, we deduce that V and $V[-1]$ are cofibrant. \square

Proposition 3.20. *There is a cofibrantly generated model structure on $FDGAlg_A(R)$ (resp. $FcAlg_A(R)$), with fibrations and weak equivalences as in Definition 3.17.*

Proof. The forgetful functor $FDGAlg_A(R) \rightarrow FDGMod_A(R)$ (resp. $FcAlg_A(R) \rightarrow FcMod_A(R)$) has a left adjoint. We may apply this to [Hir] Theorem 11.3.2 to obtain a cofibrantly generated model structure from that given in Lemma 3.18. \square

3.3.2. Lie algebras.

Definition 3.21. Define the opposite category $F\hat{\mathcal{N}}_A(R)^{\text{opp}}$ to consist of R -representations in ind-conilpotent Lie coalgebras C over A , equipped with an exhaustive increasing filtration $J_0C \subset J_1C \subset \dots$, of C as an (R, A) -module, with the property that $\nabla(J_r C) \subset \sum_{m+n=r} (J_m C) \otimes (J_n C)$, for ∇ the cobracket. Morphisms are required to respect the filtration.

Similarly, $Fdg\hat{\mathcal{N}}_A(R)$ is opposite to the category of R -representations in non-negatively filtered ind-conilpotent \mathbb{N}_0 -graded cochain Lie coalgebras over A . $Fs\hat{\mathcal{N}}_A(R)$ is the category of simplicial objects in $F\hat{\mathcal{N}}_A(R)$. When $A = \mathbb{Q}_l$, we will usually drop the subscript A .

Proposition 3.22. *There is a closed model structure on $Fdg\hat{\mathcal{N}}_A(R)$ (resp. $Fs\hat{\mathcal{N}}_A(R)$) in which a morphism $f: \mathfrak{g} \rightarrow \mathfrak{h}$ is a fibration or a weak equivalence whenever the underlying map $f^\vee: \mathfrak{h}^\vee \rightarrow \mathfrak{g}^\vee$ in $FDGMod_A(R)$ (resp. $FcMod_A(R)$) is a cofibration or a weak equivalence.*

Proof. The proof of [Pri3] Lemma 5.9 carries over to this context. \square

3.3.3. Equivalences.

Definition 3.23. Define $FcAlg(R)_{00}$ (resp. $FDGAlg(R)_{00}$) to be the full subcategory of $FcAlg_A(R)$ (resp. $FDGAlg_A(R)$) consisting of objects B with $B^0 = \mathbb{Q}_l$. Let $FcAlg(R)_0$ (resp. $FDGAlg(R)_0$) be the full subcategory consisting of objects weakly equivalent to objects of $FcAlg(R)_{00}$ (resp. $FDGAlg(R)_{00}$). Let $\text{Ho}(FcAlg(R))_0$ (resp. $\text{Ho}(FDGAlg(R))_0$) be the full subcategory of $\text{Ho}(FcAlg(R))$ (resp. $\text{Ho}(FDGAlg(R))$)

on objects $FcAlg(R)_0$ (resp. $FDGAlg(R)_0$). Denote the opposite category to $FcAlg(R)_{00}$ by $FsAff(R)_{00}$, etc.

Definition 3.24. Given $\mathfrak{g} \in Fs\hat{\mathcal{N}}(R)$, we define $\bar{W}\mathfrak{g} \in FsAff(R)$ by

$$(\bar{W}\mathfrak{g})(B) := \bar{W}(\exp(\text{Hom}_{F\text{Mod}(R)}(\mathfrak{g}^\vee, (B)))) \in \mathbb{S},$$

for $B \in \text{Alg}_A(R)$, and \bar{W} the classifying space functor, as in [GJ] §V.4, and \exp denotes exponentiation of a pro-nilpotent Lie algebra to give a Lie group.

Observe that this functor is continuous, and denote its left adjoint by $G : FsAff(R) \rightarrow Fs\hat{\mathcal{N}}(R)$.

Definition 3.25. Define functors $FdgAff(R) \xrightleftharpoons[\bar{W}]{G} Fdg\hat{\mathcal{N}}(R)$ as follows. For $\mathfrak{g} \in$

$Fdg\hat{\mathcal{N}}(R)$, the Lie bracket gives a linear map $\bigwedge^2 \mathfrak{g} \rightarrow \mathfrak{g}$. Write Δ for the dual $\Delta : \mathfrak{g}^\vee \rightarrow \bigwedge^2 \mathfrak{g}^\vee$, which respects the filtration. This is equivalent to a map $\Delta : \mathfrak{g}^\vee[-1] \rightarrow \text{Symm}^2(\mathfrak{g}^\vee[-1])$, and we define

$$O(\bar{W}\mathfrak{g}) := \text{Symm}(\mathfrak{g}^\vee[-1])$$

to be the graded polynomial ring on generators $\mathfrak{g}^\vee[-1]$, with a derivation defined on generators by $D := d + \Delta$. The Jacobi identities ensure that $D^2 = 0$.

We define G by writing $\sigma B[1]$ for the brutal truncation (in non-negative degrees) of $B[1]$, and setting

$$G(B)^\vee = \text{CoLie}(\sigma B[1]),$$

the free filtered graded Lie coalgebra over \mathbb{Q}_l , with differential similarly defined on cogenerators by $D := d + \mu$, μ here being the product on B . Note also that $G(B)$ is cofibrant for all B .

Definition 3.26. Define $\bar{G} : \text{Ho}(FsAff(R))_0 \rightarrow Fs\mathcal{M}(R)$ (resp. $\bar{G} : \text{Ho}(FdgAff(R))_0 \rightarrow Fdg\mathcal{M}(R)$) by assigning to each $X \in \text{Ho}(FsAff(R))_0$ (resp. $X \in \text{Ho}(FdgAff(R))_0$) a weakly equivalent object $X' \in FsAff(R)_{00}$ (resp. $X' \in FdgAff(R)_{00}$), and setting

$$\bar{G}(X) := G(X').$$

Definition 3.27. Define the category $Fs\mathcal{M}(R)$ (resp. $Fdg\mathcal{M}(R)$) to have the fibrant objects of $Fs\hat{\mathcal{N}}(R)$ (resp. $Fdg\hat{\mathcal{N}}(R)$), with morphisms given by

$$\begin{aligned} \text{Hom}_{Fs\mathcal{M}(R)}(\mathfrak{g}, \mathfrak{h}) &= \text{Hom}_{\text{Ho}(Fs\hat{\mathcal{N}}(R))}(\mathfrak{g}, \mathfrak{h}) / \exp(\mathfrak{h}_0^R), \\ \text{Hom}_{Fdg\mathcal{M}(R)}(\mathfrak{g}, \mathfrak{h}) &= \text{Hom}_{\text{Ho}(Fdg\hat{\mathcal{N}}(R))}(\mathfrak{g}, \mathfrak{h}) / \exp(\mathfrak{h}_0^R), \end{aligned}$$

where \mathfrak{h}_0^R is the Lie algebra $\text{Hom}_{F\text{Mod}(R)}(\mathfrak{h}_0^\vee, \mathbb{Q}_l)$, acting by conjugation on the set of homomorphisms.

Theorem 3.28. *There is the following commutative diagram of equivalences of categories:*

$$\begin{array}{ccc} \text{Ho}(FdgAff(R))_0 & \xrightleftharpoons[\mathbb{R}(\text{Spec } D^*)]{\text{Spec } D} & \text{Ho}(FsAff(R))_0 \\ \bar{G} \downarrow \uparrow \bar{W} & & \bar{G} \downarrow \uparrow \bar{W} \\ Fdg\mathcal{M}(R) & \xrightleftharpoons[N]{\mathbb{L}N^*} & Fs\mathcal{M}(R),, \end{array}$$

where N denotes normalisation, and D is denormalisation, noting that $\mathbb{L}N^*\bar{G} = N^*\bar{G}$, since everything in the image of \bar{G} is cofibrant.

Proof. The proof of [Pri3] Corollary 4.41 carries over to this context, making use of Lemma 3.19, which implies that everything in the image of \bar{W} is fibrant, as are all objects of $Fdg\hat{\mathcal{N}}(R)$ and $Fs\hat{\mathcal{N}}(R)$ \square

Although we do not have a precise analogue of this result for $\mathrm{Ho}(Fdg\mathrm{Aff}_A(R))$, we have the following:

Lemma 3.29. *Given $X \in \mathrm{Ho}(Fdg\mathrm{Aff}(R))_0$ and $\mathfrak{g} \in Fdg\hat{\mathcal{N}}(R)$,*

$$\mathrm{Hom}_{\mathrm{Ho}(Fdg\mathrm{Aff}_A(R))}(X \otimes A, \bar{W}\mathfrak{g} \otimes A) \cong \mathrm{Hom}_{\mathrm{Ho}(Fdg\hat{\mathcal{N}}_A(R))}(\bar{G}(X) \hat{\otimes} A, \mathfrak{g} \hat{\otimes} A) / \exp(\mathfrak{g}_0^R \hat{\otimes} A).$$

Proof. The proof of [Pri3] Proposition 3.48 adapts to this context. \square

Definition 3.30. We say that a filtered cochain algebra $(B, J) \in FDG\mathrm{Alg}_A(R)$ is quasi-formal if it is weakly equivalent in $FDG\mathrm{Alg}_A(R)$ to $\mathcal{J}E_1^{*,*}(B)$ (as in Definition 3.16). We say that a filtered homotopy type is quasi-formal if its associated cochain algebra is so.

3.3.4. Minimal models. Let $FDG\mathrm{Rep}(R) = FDG\mathrm{Mod}_{\mathbb{Q}_l}(R)$ be the category of non-negatively graded filtered complexes of R -representations.

Definition 3.31. We say that $M \in FDG\mathrm{Rep}(R)$ is minimal if $d(J_m M) \subset J_{m-1} M$ for all m .

Lemma 3.32. *For any $V \in FDG\mathrm{Rep}(R)$, there exists a quasi-isomorphic filtered sub-object $M \hookrightarrow V$, for M minimal.*

Proof. We prove this by induction on the filtration. Assume that we have constructed a filtered quasi-isomorphism $J_m f : J_m M \hookrightarrow J_m V$ (for $m = -1$, this is trivial). Pick a basis v_α for $H^*(\mathrm{gr}_{m+1}^J V)$, and lift v_α to $v'_\alpha \in J_{m+1} V$. Thus $dv'_\alpha \in J_m V$, and $[dv'_\alpha] = 0 \in H^*(J_m V / J_m M) = 0$. This means that $dv'_\alpha \in J_m M + dJ_m V$. Choose $u_\alpha \in J_m V$ such that $dv'_\alpha - du_\alpha \in J_m M$, and set $\tilde{v}_\alpha := v'_\alpha - u_\alpha$.

Now, $[\tilde{v}_\alpha] = v_\alpha \in H^*(\mathrm{gr}_{m+1}^J V)$, so define

$$J_{m+1} M := J_m M \oplus \langle v_\alpha \rangle_\alpha;$$

this has the properties that $dJ_{m+1} M \subset J_m M$ and $H^*(\mathrm{gr}_{m+1}^J M) \cong H^*(\mathrm{gr}_{m+1}^J V)$, as required. \square

Definition 3.33. We say that a cofibrant object $\mathfrak{m} \in Fdg\hat{\mathcal{N}}(R)$ (resp. $Fs\hat{\mathcal{N}}(R)$) is minimal if $(\mathfrak{m}/[\mathfrak{m}, \mathfrak{m}])^\vee$ (resp. $N(\mathfrak{m}/[\mathfrak{m}, \mathfrak{m}])^\vee$) is minimal in the sense of Definition 3.31.

Proposition 3.34 (Minimal models). *Every weak equivalence class in $Fdg\hat{\mathcal{N}}(R)$ (resp. $fs\hat{\mathcal{N}}(R)$) has a minimal element \mathfrak{m} , unique up to non-unique isomorphism.*

Proof. Similar to [Pri3] Proposition 1.16. \square

3.3.5. Outer automorphisms.

Definition 3.35. Given $u \in Fs\hat{\mathcal{N}}(R)$, let $G = \exp(u) \rtimes R$, and define the group presheaf of filtered outer automorphisms by

$$\mathrm{Out}_J(G)(A) := \{(f, \theta) : f \in \mathrm{Aut}(R)(A), \theta \in \mathrm{Iso}_{F\mathcal{M}_A(R)}(U \hat{\otimes} A, f^\# U \hat{\otimes} A)\}.$$

Define $\mathrm{ROut}_J(G) := \ker(\mathrm{Out}(G) \rightarrow \mathrm{Aut}(R))$.

Definition 3.36. Given $V \in \text{Rep}(R)$ and $\mathfrak{g} \in Fs\hat{\mathcal{N}}(R)$, define the spectral sequence ${}_J\mathcal{E}_*^{*,*}(R \ltimes \exp(\mathfrak{g}), V)$ to be the cohomology spectral sequence of the filtered complex

$$O(\bar{W}\mathfrak{g}) \otimes^R V,$$

for $J_0V = V$. Thus ${}_J\mathcal{E}_1^{a,b}(R \ltimes \exp(\mathfrak{g}), V) = H^{a+b}(\text{Gr}_{-a}^J O(\bar{W}\mathfrak{g}) \otimes^R V)$.

Lemma 3.37. Assume that G is as above, and let $\mathfrak{m} \in Fs\hat{\mathcal{N}}(R)$ be a minimal model for $R_{\text{u}}(G)$. If $H^i(G, V)$ is finite-dimensional for all i and all finite-dimensional irreducible R -representations V , then the group presheaves

$$\text{Aut}_{Fs\hat{\mathcal{N}}(R)}(\mathfrak{m}) \xrightarrow{\alpha} \text{ROut}_J(G) \xrightarrow{\beta} \prod_{a,b} \text{Aut}_R({}_J\mathcal{E}_1^{a,b}(G, O(R)))$$

are all pro-algebraic groups, α, β both have pro-unipotent kernels, and β is surjective.

Proof. Similar to [Pri3] Theorem 5.13. \square

3.3.6. Examples.

Definition 3.38. Given $B^\bullet \in D\text{GAlg}_A(R)$, we define the good truncation τ_* on B by

$$(\tau_m B)^n := \begin{cases} B^n & n < m \\ Z^m(B) & n = m \\ 0 & n > m. \end{cases}$$

Observe that $(B^\bullet, \tau) \in FD\text{GAlg}_A(R)$.

Definition 3.39. Given a bicosimplicial algebra $B^{\bullet,\bullet} \in cc\text{Alg}_A(R)$, we define the associated filtered cosimplicial algebra $(\tau_0'' B \leq \tau_1'' B \leq \dots) \in Fc\text{Alg}_A(R)$ by

$$(\tau_m'' B)^n = (D\tau_m \mathbb{L} D^* B^{n,\bullet})^n,$$

for $D, \mathbb{L} D^*$ as in Theorem 3.2. Observe that there is a canonical quasi-isomorphism $\text{diag } B^{\bullet,\bullet} \rightarrow \tau_\infty'' B^\bullet$.

In practice, the only filtered homotopy types which we will encounter come from morphisms of spaces:

Definition 3.40. Given an algebraic variety X and a ind-constructible l -adic sheaf \mathbb{V} on X , recall (e.g. from [Pri2] Definition 2.3) that there is a natural cosimplicial complex

$$\mathcal{C}_{\text{ét}}^\bullet(\mathbb{V})$$

of l -adic sheaves on \mathbb{V} , with the property that $\Gamma(X, \mathcal{C}_{\text{ét}}^\bullet(\mathbb{V})) = \mathbf{C}_{\text{ét}}^\bullet(X, \mathbb{V})$, the Godement resolution (as in Remark 2.24). This construction respects tensor products.

Lemma 3.41. To any morphism $j : Y \rightarrow X$ of algebraic varieties, and any \mathbb{Q}_l -sheaf \mathcal{S} of algebras on Y as in Definition 3.40, there is associated a canonical filtered homotopy type $\mathbf{C}_{\text{ét}}^\bullet(j, \mathcal{S}) \in \text{Ho}(Fc\text{Alg}_{\mathbb{Q}_l})$, with the property that ${}_J\mathcal{E}_*^{*,*} \mathbf{C}_{\text{ét}}^\bullet(j, \mathcal{S})$ is the Leray spectral sequence

$${}_J\mathcal{E}_1^{a,b} \mathbf{C}_{\text{ét}}^\bullet(j, \mathcal{S}) = H^{2a+b}(X, R^{-a} j_* \mathcal{S}) \implies H^{a+b}(Y, \mathcal{S}).$$

The associated unfiltered homotopy type is canonically weakly equivalent to $\mathbf{C}_{\text{ét}}^\bullet(Y, \mathcal{S})$.

Proof. We have a \mathbb{Q}_l -sheaf $j_*\mathcal{C}_{\text{ét}}^\bullet(\mathcal{S})$ of cosimplicial algebras on X , and hence a bicosimplicial algebra

$$\mathbf{C}_{\text{ét}}^\bullet(X, j_*\mathcal{C}_{\text{ét}}^\bullet(\mathcal{S})).$$

Now, set

$$J_n\mathbf{C}_{\text{ét}}^\bullet(j, \mathcal{S}) = \tau_n''\mathbf{C}_{\text{ét}}^\bullet(X, j_*\mathcal{C}_{\text{ét}}^\bullet(\mathcal{S})) = \text{diag } \mathbf{C}_{\text{ét}}^\bullet(X, D\tau_n\mathbb{L}D^*j_*\mathcal{C}_{\text{ét}}^\bullet(\mathcal{S})),$$

as in Definition 3.39, with $\mathbf{C}_{\text{ét}}^\bullet(X, j_*\mathcal{C}_{\text{ét}}^\bullet(\mathcal{S})) \rightarrow J_\infty\mathbf{C}_{\text{ét}}^\bullet(j, \mathcal{S})$ a quasi-isomorphism..

Finally, observe that there is a quasi-isomorphism

$$\mathbf{C}_{\text{ét}}^\bullet(Y, \mathcal{S}) = \Gamma(X, j_*\mathcal{C}_{\text{ét}}^\bullet(\mathcal{S})) \rightarrow \text{diag } \mathbf{C}_{\text{ét}}^\bullet(X, j_*\mathcal{C}_{\text{ét}}^\bullet(\mathcal{S})),$$

and that $\text{gr}_n^\tau j_*\mathcal{C}_{\text{ét}}^\bullet(\mathcal{S})$ is quasi-isomorphic to $\text{R}^n j_*\mathcal{S}$. \square

Remark 3.42. There is a similar statement for filtrations on homotopy types coming from morphisms of topological spaces, using Čech resolutions instead of Godement resolutions.

Definition 3.43. Given a morphism $j : Y \rightarrow X$ of algebraic varieties and a Zariski-dense continuous map

$$\rho : \widehat{\pi_f(X)} \rightarrow R(\mathbb{Q}_l)$$

define the filtered homotopy type $(Y^{\rho, \text{Mal}}, j)$ to correspond to $\mathbf{C}_{\text{ét}}^\bullet(j, \mathbb{O}(R)) \in \text{FcAlg}(R)$.

4. ALGEBRAIC GALOIS ACTIONS

4.1. Weight decompositions. By a weight decomposition, we will mean an algebraic action of the group \mathbb{G}_m . A weight decomposition on a vector space V is equivalent to a decomposition $V = \bigoplus_{n \in \mathbb{Z}} \mathcal{W}_n V$, given by $\lambda \in \mathbb{G}_m$ acting as λ^n on $\mathcal{W}_n V$.

Fix a prime p , which need not differ from l . Let \mathbb{Z}^{alg} be the pro-algebraic group over \mathbb{Q}_l parametrising \mathbb{Z} -representations. It takes the form $\mathbb{Z}^{\text{alg}} = \mathbb{G}_a \times \mathbb{Z}^{\text{red}}$, where \mathbb{Z}^{red} is its reductive quotient.

Definition 4.1. Given $n \in \mathbb{Z}$ and a power q of p , recall that an element $\alpha \in \bar{\mathbb{Q}}_l$ is said to be pure of weight n if it is algebraic and all of its complex conjugates have absolute value $q^{n/2}$.

Let M_q be the quotient of \mathbb{Z}^{red} whose representations ρ correspond to semisimple \mathbb{Z} -representations for which the eigenvalues of $\rho(1)$ are all of integer weight with respect to q . Such representations are called mixed.

Observe that every M_q -representation decomposes into “pure” representations, in which all eigenvalues have the same weight. There is thus a canonical map $\mathbb{G}_m \rightarrow M_q$ given by $\lambda \in \mathbb{G}_m$ acting as λ^n on a pure representation of weight n .

Definition 4.2. Define P_q to be the quotient of M_q whose representations are pure of weight 0.

Definition 4.3. Given $n \in \mathbb{Z}$, an embedding $\iota : \bar{\mathbb{Q}}_l \rightarrow \mathbb{C}$ and a power q of p , recall that an element $\alpha \in \bar{\mathbb{Q}}_l$ is said to be ι -pure of weight n if $|\iota(\alpha)| = q^{n/2}$.

Let $M_{q, \iota}$ be the quotient of \mathbb{Z}^{red} whose representations ρ correspond to semisimple \mathbb{Z} -representations for which the eigenvalues of $\rho(1)$ are all of integer ι -weight. Note that M_q is a quotient of $M_{\iota, q}$.

Observe that there is a canonical map $\mathbb{G}_m \rightarrow M_{\iota, q}$ given by $\lambda \in \mathbb{G}_m$ acting as λ^n on an ι -pure representation of weight n , and that this induces the map $\mathbb{G}_m \rightarrow M_q$ above.

Definition 4.4. Define $P_{\iota,q}$ to be the quotient of $M_{\iota,q}$ whose representations are pure of ι -weight 0.

Definition 4.5. Given a pro-algebraic group G , let G^0 be the connected component of the identity; if \hat{G} is the maximal pro-finite quotient of G (parametrising representations with finite monodromy), then $G^0 = \ker(G \rightarrow \hat{G})$.

Lemma 4.6. *If Γ is a pro-discrete group, then we may identify*

$$\Gamma^{\text{alg},0} = \varprojlim_{\Delta} \Delta^{\text{alg}},$$

where Δ runs over $\Delta \triangleleft \Gamma$ open of finite index.

Proof. First note that $\widehat{\Gamma^{\text{alg}}} = \hat{\Gamma}$. The exact sequence $\Delta \rightarrow \Gamma \rightarrow \Gamma/\Delta \rightarrow 1$ gives an exact sequence $(\Delta)^{\text{alg}} \xrightarrow{\alpha} \Gamma^{\text{alg}} \rightarrow \Gamma/\Delta \rightarrow 1$. It suffices to show that α is injective. This follows from the observation that every finite-dimensional Δ -representation V embeds into a finite-dimensional Γ -representation $\text{Ind}_{\Delta}^{\Gamma} V$. \square

Thus if F is a generator for \mathbb{Z} , then representations of $\mathbb{Z}^{\text{alg},0}$ are sums of F^r -representations, with morphisms commuting locally with sufficiently high powers of F .

Observe that we have commutative diagrams

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{r} & \mathbb{Z} \\ \downarrow & & \downarrow \\ M_{qr} & \longrightarrow & M_q. \end{array}$$

Any \mathbb{Z} -representation with finite monodromy is pure of weight 0, giving a map

$$P_p \rightarrow \hat{\mathbb{Z}}.$$

Also note that $M_{qr} = \ker(M_q \rightarrow \mathbb{Z}/r\mathbb{Z})$.

Lemma 4.7. *Observe that*

$$M^0 := M_p^0 = \varprojlim M_{p^r}, \quad P^0 := P_p^0 = \varprojlim P_{p^r};$$

there are quotient maps $\mathbb{Z}^{\text{red},0} \twoheadrightarrow M^0 \twoheadrightarrow P^0$. There are similar results for $M_{\iota}^0 := M_{\iota,p}^0$, $P_{\iota}^0 := P_{\iota,p}^0$.

Definition 4.8. We say that a representation of $\mathbb{Z}^{\text{alg},0}$ is mixed (resp. pure of weight 0, resp. ι -mixed with integral weights, resp. ι -pure) if the action of $\mathbb{Z}^{\text{red},0} \triangleleft \mathbb{Z}^{\text{alg},0}$ factors through M^0 (resp. P^0 , resp. M_{ι}^0 , resp. P_{ι}^0).

Lemma 4.9. *Observe that the canonical maps $\mathbb{G}_m \rightarrow M_q$ are compatible, giving $\mathbb{G}_m \rightarrow M^0$, with trivial image in P^0 . Similarly, we have $\mathbb{G}_m \rightarrow M_{\iota}^0$, with trivial image in P_{ι}^0 .*

4.1.1. *Slope decompositions.*

Definition 4.10. Define the proalgebraic group $\widetilde{\mathbb{G}_m}$ to be the inverse limit of the étale universal covering system of \mathbb{G}_m . This is the inverse system $\{G_r\}_{r \in \mathbb{N}}$ with $G_r = \mathbb{G}_m$ and morphisms $G_{sr} \xrightarrow{[s]} G_r$, for $s \in \mathbb{N}$.

Lemma 4.11. *The category of $\widetilde{\mathbb{G}_m}$ -representations is canonically equivalent to the category of \mathbb{Q} -graded vector spaces.*

Proof. A representation of \mathbb{G}_m is equivalent to a \mathbb{Z} -grading. Given a finite-dimensional vector space V with a \mathbb{Q} -grading $V = \bigoplus V_\lambda$, let d be the lowest common multiple of the denominators of the set $\{\lambda \in \mathbb{Q} : V_\lambda \neq 0\}$. Then $V = \bigoplus_{n \in \mathbb{Z}} V_{n/d}$, giving a \mathbb{G}_m -action on V . If we regard this copy of \mathbb{G}_m as G_d , this defines a $\widehat{\mathbb{G}_m}$ -action. \square

Now assume that $p = l$.

Definition 4.12. Given a power q of p , normalise the p -adic valuation v on $\bar{\mathbb{Q}}_p$ by $v(q) = 1$. Define the slope of $\alpha \in \bar{\mathbb{Q}}_p$ to be $v(\alpha) \in \mathbb{Q}$.

Lemma 4.13. *There is a canonical morphism $\widehat{\mathbb{G}_m} \rightarrow \mathbb{Z}^{\text{red}}$, corresponding to the functor sending a \mathbb{Z} -representation V to a slope decomposition $\bigoplus V_\lambda$.*

Proof. Let F be the canonical generator for \mathbb{Z} . Given a finite-dimensional semisimple \mathbb{Z} -representation V , we may decompose $V \otimes_{\mathbb{Q}_p} \bar{\mathbb{Q}}_p$ into F -eigenspaces, and hence take a decomposition by slopes of the eigenvalues. Since conjugates in $\bar{\mathbb{Q}}_p$ have the same slope, this descends to a slope decomposition $V = \bigoplus_{\lambda \in \mathbb{Q}} V_\lambda$, as required. \square

4.2. Potentially unramified actions. Fix a prime $p \neq l$, and take a local field K , with finite residue field k of characteristic p . Denote the canonical generator of $\text{Gal}(\bar{k}/k)$ by F , the Frobenius automorphism, and set $\Gamma := \text{Gal}(\bar{K}/K) \times_{\text{Gal}(\bar{k}/k)} \langle F \rangle$ — this is a topological group whose profinite completion is $\text{Gal}(\bar{K}/K)$.

Let $\mathcal{G} := \Gamma^{\text{alg}}$, the pro-algebraic completion of Γ over \mathbb{Q}_l , and note that $\text{Gal}(\bar{K}/K)^{\text{alg}}$ is a quotient of \mathcal{G} .

Definition 4.14. Say that a finite-dimensional continuous \mathbb{Q}_l -representation of Γ is potentially unramified if there exists a finite extension K'/K for which the action of $\text{Gal}(\bar{K}/K') \cap \Gamma$ is unramified. Say that an arbitrary \mathbb{Q}_l -representation of Γ is potentially unramified if it is a direct sum of finite-dimensional potentially unramified representations.

These form a neutral Tannakian category; let \mathcal{G}^{pnr} be the corresponding pro-algebraic group. Since $\text{Rep}(\mathcal{G}^{\text{pnr}})$ is a full subcategory of $\text{Rep}(\mathcal{G})$ closed under subobjects, \mathcal{G}^{pnr} is a quotient of \mathcal{G} .

Lemma 4.15. *There is a canonical morphism $\mathbb{Z}^{\text{alg},0} \rightarrow \mathcal{G}^{\text{pnr}}$ of pro-algebraic groups, for $\mathbb{Z}^{\text{alg},0}$ as in Definition 4.6. The composition $\mathbb{Z}^{\text{alg},0} \rightarrow \langle F \rangle^{\text{alg}}$ is the usual embedding $\mathbb{Z}^{\text{alg},0} \hookrightarrow \mathbb{Z}^{\text{alg}}$.*

Proof. If $(K')^{\text{nr}}$ denotes the maximal unramified extension of K' with residue field k' and k'/k is of degree r , then $F^r \in \text{Gal}((K')^{\text{nr}}/K')$. Every finite-dimensional potentially unramified Γ -representation therefore has an action of F^r for some r , as required. Equivalently, observe that $\mathcal{G}^{\text{pnr}} \cong \text{Gal}(\bar{K}/K) \times_{\text{Gal}(\bar{k}/k)} \text{Gal}(\bar{k}/k)^{\text{alg}}$, so $\mathcal{G}^{\text{pnr},0} = \text{Gal}(\bar{k}/k)^{\text{alg},0} \cong \mathbb{Z}^{\text{alg},0}$. \square

Definition 4.16. We say that a representation of \mathcal{G}^{pnr} is mixed (resp. pure of weight 0) if the resulting action of $\mathbb{Z}^{\text{alg},0}$ is so.

4.3. Potentially crystalline actions. Now let $l = p$, and take a local field K , with finite residue field k of order $q = p^f$. Let $\mathcal{G} := \text{Gal}(\bar{K}/K)^{\text{alg}}$, the pro-algebraic completion of $\text{Gal}(\bar{K}/K)$ over \mathbb{Q}_p . Let $W := W(k)$, with fraction field K_0 , and let σ denote the unique lift of Frobenius $\Phi \in \text{Gal}(\bar{k}/\mathbb{F}_p)$ to $\sigma \in \text{Gal}(K_0^{\text{nr}}/\mathbb{Q}_p)$, for K_0^{nr} the maximal unramified extension of K_0 . Note that the Frobenius of the previous section is $F = \Phi^f$.

Definition 4.17. Say that a finite-dimensional continuous $\mathrm{Gal}(\bar{K}/K)$ -representation over \mathbb{Q}_p is potentially crystalline if there exists a finite extension K'/K for which the action of $\mathrm{Gal}(\bar{K}/K')$ is crystalline. Say that an arbitrary \mathbb{Q}_l -representation of $\mathrm{Gal}(\bar{K}/K)$ is potentially crystalline if it is a direct sum of finite-dimensional potentially crystalline representations. Note that since unramified representations are automatically crystalline, all potentially unramified representations are potentially crystalline.

These form a neutral Tannakian category; let $\mathcal{G}^{\mathrm{pcris}}$ be the corresponding pro-algebraic group. Since $\mathrm{Rep}(\mathcal{G}^{\mathrm{pnr}})$ is a full subcategory of $\mathrm{Rep}(\mathcal{G})$ closed under subobjects, $\mathcal{G}^{\mathrm{pnr}}$ is a quotient of \mathcal{G} .

In [Fon], Fontaine defined a ring of periods $B_{\mathrm{cris}} := B_{\mathrm{cris}}(W(k))$ over \mathbb{Q}_p , equipped with a Hodge filtration and actions of $\mathrm{Gal}(\bar{K}/K)$ and Frobenius, and used it to characterise crystalline representations (adapted in Lemma 4.19 below).

Definition 4.18. Given a finite-dimensional $\mathrm{Gal}(\bar{K}/K)$ -representation U , set

$$D_{\mathrm{pcris}}(U) := \varinjlim (U \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{\mathrm{Gal}(\bar{K}/L)},$$

for L ranging over all finite extensions of K contained in \bar{K} . For an arbitrary algebraic $\mathrm{Gal}(\bar{K}/K)$ -representation U , set

$$D_{\mathrm{pcris}}(U) := \varinjlim D_{\mathrm{pcris}}(U_\alpha),$$

for U_α running over all finite-dimensional subrepresentations.

Lemma 4.19. *Recall that a $\mathrm{Gal}(\bar{K}/K)$ -representation U is potentially crystalline if and only if the canonical map*

$$D_{\mathrm{pcris}}(U) \otimes_{K_0^{\mathrm{nr}}} B_{\mathrm{cris}} \rightarrow (U \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})$$

is an isomorphism.

Observe that $\mathrm{Spec} B_{\mathrm{cris}}$ is an affine \mathcal{G} -scheme over $\mathrm{Spec} \mathbb{Q}_p$, and that the coarse quotient $\mathrm{Spec} B_{\mathrm{cris}}/\mathcal{G}^0 = \mathrm{Spec} K_0^{\mathrm{nr}}$.

Corollary 4.20. *An action of \mathcal{G} on an affine \mathbb{Q}_p -scheme Y factors through $\mathcal{G}^{\mathrm{pcris}}$ if and only if the surjective map*

$$Y \times_{\mathbb{Q}_p} \mathrm{Spec} B_{\mathrm{cris}} \rightarrow (Y \times^{\mathcal{G}^0} \mathrm{Spec} B_{\mathrm{cris}}) \times_{K_0^{\mathrm{nr}}} \mathrm{Spec} B_{\mathrm{cris}}$$

is an isomorphism.

Proof. Note that $D_{\mathrm{pcris}}(V) = V \otimes^{\mathcal{G}^0} B_{\mathrm{cris}}$, so $D_{\mathrm{pcris}}(O(Y)) = O(Y \times^{\mathcal{G}^0} \mathrm{Spec} B_{\mathrm{cris}})$. \square

Corollary 4.21. *An action of \mathcal{G} on an affine \mathbb{Q}_p -scheme Y factors through $\mathcal{G}^{\mathrm{pcris}}$ if and only if there exists an affine K_0^{nr} -scheme Z , with*

$$Y \times_{\mathbb{Q}_p} \mathrm{Spec} B_{\mathrm{cris}} \cong Z \times_{K_0^{\mathrm{nr}}} \mathrm{Spec} B_{\mathrm{cris}}$$

a \mathcal{G}^0 -equivariant map (for trivial \mathcal{G}^0 -action on Z).

Proof. Taking $Z = Y \times^{\mathcal{G}^0} \mathrm{Spec} B_{\mathrm{cris}}$ proves necessity, since the action of $\mathrm{Gal}(\bar{K}/K)$ on Z then has pro-finite monodromy, giving a trivial \mathcal{G}^0 -action. Conversely, the expression implies that $Z = Y \times^{\mathcal{G}^0} \mathrm{Spec} B_{\mathrm{cris}}$, so Lemma 4.19 is satisfied. \square

4.3.1. *Frobenius actions.* Although we do not have a canonical map $\mathbb{Z}^{\text{alg},0} \rightarrow \mathcal{G}^{\text{pcris}}$, there is something nearly as strong:

Lemma 4.22. *There is a canonical morphism*

$$\mathbb{Z}^{\text{alg},0} \otimes_{\mathbb{Q}_p} B_{\text{cris}}^\sigma \rightarrow \mathcal{G}^{\text{pcris}} \otimes_{\mathbb{Q}_p} B_{\text{cris}}^\sigma$$

of affine group schemes over the σ -invariant subring B_{cris}^σ of B_{cris} .

Proof. Given $U \in \text{FDRep}(\mathcal{G}^{\text{pcris}})$, U is crystalline over K' for some finite extension K'/K with residue field k' . If $|k'/k| = r$ and $q = p^f$, then ϕ^{fr} is a K'_0 -linear endomorphism of $D_{\text{cris},K'}(U)$. We may extend this K'_0 -linearly to give an automorphism F_r of $D_{\text{pcris}}(U)$ (note that $F_r \neq \phi^{fr}$, the latter being σ -semilinear).

Now, observe that $D_{\text{pcris}}(U)$ is a direct sum of finite-dimensional F_r -representations over \mathbb{Q}_p , since $D_{\text{cris},K'}(U)$ is finite-dimensional over K' , and hence over \mathbb{Q}_p . This gives us a σ -equivariant \mathbb{Q}_p -linear action of $\mathbb{Z}^{0,\text{alg}}$ on $D_{\text{pcris}}(U)$, and hence a σ -equivariant B_{cris}^σ -linear action on $D_{\text{pcris}}(U) \otimes_{K'_0} B_{\text{cris}} = U \otimes_{\mathbb{Q}_p} B_{\text{cris}}^\sigma$. We now take the ϕ -invariant subspace, giving a $\mathbb{Z}^{0,\text{alg}} \otimes_{\mathbb{Q}_p} B_{\text{cris}}^\sigma$ -action on $U \otimes_{\mathbb{Q}_p} B_{\text{cris}}^\sigma$. This is canonical, hence functorial and compatible with tensor products, so by Tannakian duality gives the required map. \square

Definition 4.23. We say that a potentially crystalline representation U is mixed (resp. pure, resp. ι -mixed with integral weights, resp. ι -pure) if the action of $\mathbb{Z}^{\text{alg},0} \otimes B_{\text{cris}}^\sigma$ on $U \otimes B_{\text{cris}}^\sigma$ factors through M_q (resp. P_q , resp. $M_{\iota,q}$, resp. $P_{\iota,q}$). This is equivalent to saying that the action of \mathbb{Z} on $D_{\text{pcris}}(U)$ is mixed (resp. pure, resp. ι -mixed with integral weights, resp. ι -pure).

We have the following analogue of a slope decomposition:

Lemma 4.24. *There is a canonical morphism $\widetilde{\mathbb{G}}_m \rightarrow \mathcal{G}^{\text{pcris}} \otimes_{\mathbb{Q}_p} B_{\text{cris}}^\sigma$ of affine group schemes over B_{cris}^σ , for $\widetilde{\mathbb{G}}_m$ as in definition 4.10.*

Proof. Combine Lemma 4.13 with Lemma 4.22. \square

5. VARIETIES OVER FINITE FIELDS

Fix a variety X_k over a finite field k , of order q prime to l . Let $X := X_k \otimes_k \bar{k}$, for \bar{k} the algebraic closure of k . There is a Frobenius endomorphism on X , and hence on the pro-simplicial set $X_{\text{ét}}$, and on its algebraisation $(X_{\text{ét}})^{\text{alg}}$. The purpose of this section is to describe this action as far as possible.

5.1. Algebraising the Weil groupoid. The morphism $X \rightarrow X_k$ gives a map of groupoids $\alpha : \pi_f^{\text{ét}} X \rightarrow \pi_f^{\text{ét}}(X_k)$. Similarly, there is a map $\pi_f^{\text{ét}} X_k \rightarrow \pi_f^{\text{ét}} \text{Spec } k = \text{Gal}(\bar{k}/k) \cong \hat{\mathbb{Z}}$. Denote the canonical generator of $\text{Gal}(\bar{k}/k)$ by F , the Frobenius automorphism.

In constructing fundamental groupoids and étale homotopy types, we may use the same set of geometric points for both X_k and X , so assume that α is an isomorphism on objects. We then have

$$\pi_f^{\text{ét}}(X) = \pi_f^{\text{ét}}(X_k) \times_{\hat{\mathbb{Z}}} 1.$$

Definition 5.1. Define the Weil groupoid $W_f(X_k)$ by

$$W_f(X_k) := \pi_f^{\text{ét}}(X_k) \times_{\hat{\mathbb{Z}}} \mathbb{Z},$$

noting that this is a pro-discrete groupoid with discrete objects.

Definition 5.2. Define ${}^W\varpi_f^{\text{ét}}(X)$ to be the image of $\varpi_f^{\text{ét}}(X) \rightarrow W_f(X_k)^{\text{alg}}$. Thus linear representations of ${}^W\varpi_f^{\text{ét}}(X)$ correspond to \mathbb{Q}_l -local systems on X arising as subrepresentations of Weil sheaves. Note that [Pri5] Lemma 1.11 ensures that this definition is consistent with the definition of ${}^W\varpi_1(X, \bar{x})$ given in [Pri5] (as the universal object classifying continuous $W(X_0, x)$ -equivariant homomorphisms $\pi_1(X, \bar{x}) \rightarrow G(\mathbb{Q}_l)$).

Define ${}^{\text{Gal}}\varpi_f^{\text{ét}}(X)$ to be the image of $\varpi_f^{\text{ét}}(X) \rightarrow \varpi_f^{\text{ét}}(X_k)$. Representations of ${}^{\text{Gal}}\varpi_f^{\text{ét}}(X)$ correspond to \mathbb{Q}_l -local systems on X arising as subrepresentations of pull-backs of local systems on X_k . Note that ${}^{\text{Gal}}\varpi_f^{\text{ét}}(X)$ is a Frobenius-equivariant quotient of ${}^W\varpi_f^{\text{ét}}(X)$ (it is in fact the quotient on which $\hat{\mathbb{Z}}$ acts continuously).

Lemma 5.3. *The canonical action of F on ${}^W\varpi_f^{\text{ét}}(X)$ factors through a morphism*

$$\mathbb{Z}^{\text{alg}} \rightarrow \text{Aut}({}^W\varpi_f^{\text{ét}}(X))$$

of group presheaves, for \mathbb{Z}^{alg} as in §4.1.

Proof. Write $G = {}^W\varpi_f^{\text{ét}}(X), H = W_f(X_k)^{\text{alg}}$. First observe that the orbits of F in $\text{Ob } G = \text{Ob } H$ are finite, giving a map

$$\hat{\mathbb{Z}} \rightarrow \text{Aut}(\text{Ob } H).$$

Since $\hat{\mathbb{Z}}$ is pro-finite, it is a pro-algebraic group and there is a surjection $\mathbb{Z}^{\text{alg}} \rightarrow \hat{\mathbb{Z}}$.

Now, consider the group scheme

$$N := \coprod_{f \in \text{Aut}(\text{Ob}(H))} \coprod_{x \in \text{Ob}(H)} H(x, fx),$$

with multiplication given by

$$(f, \{h_x\}) \cdot (f', \{h'_x\}) = (f \cdot f', \{h_{f'x} \cdot h_x\}).$$

There is a morphism $N \rightarrow \text{Aut}(\text{Ob}(H))$ fibred in affine schemes. Thus

$$\hat{\mathbb{Z}} \times_{\text{Aut}(\text{Ob}(H))} N$$

is an affine scheme.

Now, F gives a collection of paths $F(x) \in W_f(X_k)(x, Fx)$, and thus a map

$$\mathbb{Z} \rightarrow (\hat{\mathbb{Z}} \times_{\text{Aut}(\text{Ob}(H))} N)(\mathbb{Q}_l).$$

Since the latter is an affine group scheme, this extends to a map $\mathbb{Z}^{\text{alg}} \rightarrow \hat{\mathbb{Z}} \times_{\text{Aut}(\text{Ob}(H))} N$. Finally, observe that the conjugation action of H on G gives a map

$$N \rightarrow \text{Aut}(G).$$

□

Theorem 5.4. *The action of \mathbb{Z}^{red} on ${}^W\varpi_f^{\text{ét}}(X)^{\text{red}}$ factors through P_q (see Definition 4.2).*

Proof. Since $\mathbb{Z}^{\text{alg}} = \mathbb{Z}^{\text{red}} \times \mathbb{G}_a$, this amounts to showing that the Frobenius action factors through $P \times \mathbb{G}_a$. This follows from Lafforgue's Theorem ([Laf] Theorem VII.6 and Corollary VII.8), with the details of the proof as in [Pri5] Theorem 1.14. □

5.2. Weight decompositions. Now assume that X is either smooth or proper.

Definition 5.5. Recall from [Pri3] Definition 5.15 that a weight decomposition on $G \in s\mathcal{E}(R)$ is defined to be a morphism

$$\mathbb{G}_m \rightarrow \mathrm{ROut}(G)$$

of pro-algebraic groups.

Proposition 5.6. *If we let R be any Frobenius-equivariant quotient of ${}^W\varpi_f^{\mathrm{ét}}(X)^{\mathrm{red}}$, and $\rho : \varpi_f^{\mathrm{ét}}X \rightarrow R$, then the outer Frobenius action on*

$$X_{\mathrm{ét}}^{\rho, \mathrm{Mal}}$$

is mixed, giving a canonical weight decomposition.

Proof. By Theorem 5.4, the Frobenius action on R factors through the quotient $P \times \mathbb{G}_a$ of $\mathbb{Z}^{\mathrm{alg}}$. By Corollary 3.13, the Frobenius action on $X_{\mathrm{ét}}^{\rho, \mathrm{Mal}}$ is algebraic. Since R is a $P \times \mathbb{G}_a$ -representation, $\bigoplus_{x \in \mathrm{Ob} R} \mathbb{O}(R)(x)$ is a pure Weil representation of weight 0. Deligne's Weil II theorems ([Del2] Corollaries 3.3.4 – 3.3.6) then imply that $\bigoplus_{x \in X} H^*(X, \mathbb{O}(R)(x))$ is a mixed Frobenius representation (i.e. a representation of $M \times \mathbb{G}_a$). By Lemma 3.14, we may therefore conclude that the outer action of $\mathbb{Z}^{\mathrm{red}}$ on $X_{\mathrm{ét}}^{\rho, \mathrm{Mal}}$ factors through M , i.e.

$$M \rightarrow \mathrm{Out}(X_{\mathrm{ét}}^{\rho, \mathrm{Mal}}).$$

Finally, use the map $\mathbb{G}_m \rightarrow M$ (given after Definition 4.1) to define the weight decomposition. Since R is pure of weight zero, this gives a map

$$\mathbb{G}_m \rightarrow \mathrm{ROut}(X_{\mathrm{ét}}^{\rho, \mathrm{Mal}}),$$

as required. \square

Corollary 5.7. *There are canonical weight decompositions on the homotopy groups $\varpi_n(X_{\mathrm{ét}}^{\rho, \mathrm{Mal}})$, unique up to conjugation by $\mathrm{R}_u(\varpi_f X_{\mathrm{ét}}^{\rho, \mathrm{Mal}})$.*

Proof. Apply [Pri3] Lemma 5.16. \square

Remark 5.8. We have shown that $\varpi_n(X_{\mathrm{ét}}^{\rho, \mathrm{Mal}})$ is a mixed Weil representation. In particular, this means that $\varpi_n(X_{\mathrm{ét}}^{\rho, \mathrm{Mal}}, \bar{x})$ is a mixed F_x -representation, so has a canonical weight decomposition.

Corollary 5.9. *If L is some set of primes containing l for which $\rho : (\pi_f^{\mathrm{ét}}X)^{\hat{L}} \rightarrow {}^W\varpi_f^{\mathrm{ét}}(X)^{L, \mathrm{red}}$ is good, with the homotopy groups $\varpi_n(-) := \pi_n^{\mathrm{ét}}(X^{\hat{L}}, -) \otimes_{\hat{\mathbb{Z}}} \mathbb{Q}_l$ finite-dimensional for all $n > 1$, then $\varpi_n(-)$ is a mixed Weil representation. Thus $\varpi_n(x)$ is a mixed F_x -representation, so has a canonical weight decomposition.*

Proof. Combine Proposition 5.6 and Proposition 2.44. \square

5.3. Formality. Now assume that X is smooth and proper. Deligne's Weil II theorems then imply that $\bigoplus_{x \in X} H^n(X, \mathbb{O}(R)(x))$ is pure of weight n .

Theorem 5.10. *For ρ as in Proposition 5.6, the Malcev homotopy type $X_{\mathrm{ét}}^{\rho, \mathrm{Mal}} \in s\mathcal{E}(R)$ is formal, in the sense that it corresponds (under the equivalences of [Pri3] Theorem 4.41) to the R -representation*

$$H_{\mathrm{ét}}^*(X, \mathbb{O}(R))$$

in cochain algebras. This isomorphism can be chosen to be Frobenius equivariant.

Proof. We need to construct an isomorphism $\theta : \mathrm{NR}_u(X_{\mathrm{\acute{e}t}}^{\rho, \mathrm{Mal}}) \cong \bar{G}\mathrm{H}_{\mathrm{\acute{e}t}}^*(X, \mathcal{O}(R))$ in $dg\mathcal{M}(R)$, such that $\mathrm{ad}_\theta : \mathrm{Out}(X_{\mathrm{\acute{e}t}}^{\rho, \mathrm{Mal}}) \rightarrow \mathrm{Out}(\bar{G}\mathrm{Spec} \mathrm{DH}_{\mathrm{\acute{e}t}}^*(X, \mathcal{O}(R)) \rtimes R)$ satisfies $\mathrm{ad}_\theta F = F$.

As in §3.3.4, take a minimal model \mathfrak{m} for $\mathrm{NR}_u(X_{\mathrm{\acute{e}t}}^{\rho, \mathrm{Mal}}) \in dg\hat{\mathcal{N}}(R)$. This has the property that $\mathfrak{m}_n/[\mathfrak{m}, \mathfrak{m}]_n \cong H^{n+1}(X, \mathcal{O}(R))^\vee$.

From the proof of [Pri3] Theorem 5.13, we know that

$$\mathrm{Aut}_{dg\hat{\mathcal{N}}_A(R)}(\mathfrak{m} \hat{\otimes} A) \rightarrow \mathrm{ROut}(X_{\mathrm{\acute{e}t}}^{\rho, \mathrm{Mal}})(A)$$

is a pro-unipotent extension of pro-algebraic groups. Similarly, if we write $\mathrm{Aut}(R \ltimes \mathfrak{m}, R)$ for the group of automorphisms of $R \ltimes \mathfrak{m}$ preserving the subgroup R , then the maps

$$\begin{aligned} & \mathrm{Aut}(R \ltimes \mathfrak{m}, R) \twoheadrightarrow \mathrm{Out}(X_{\mathrm{\acute{e}t}}^{\rho, \mathrm{Mal}}) \rightarrow \\ & \{(f, \alpha) : f \in \mathrm{Aut}(R), \alpha \in \mathrm{Iso}_{DGA\mathrm{lg}(R)}(H_{\mathrm{\acute{e}t}}^*(X, \mathcal{O}(R)), f^\# H_{\mathrm{\acute{e}t}}^*(X, \mathcal{O}(R)))\} \end{aligned}$$

both have pro-unipotent kernels.

We may therefore lift the Frobenius endomorphism $F \in \mathrm{Out}(X_{\mathrm{\acute{e}t}}^{\rho, \mathrm{Mal}})$ to an automorphism of $(R \ltimes \mathfrak{m}, R)$. This gives a lift of the weight decomposition $\mathbb{G}_m \rightarrow \mathrm{ROut}(X_{\mathrm{\acute{e}t}}^{\rho, \mathrm{Mal}})$ to $\mathrm{Aut}(\mathfrak{m})$.

Let $V_n := \mathcal{W}_{-n-1}\mathfrak{m}_n$, for \mathcal{W} as in §4.1; since cohomology is pure, we deduce that $V_n \rightarrow H^{n+1}(X, \mathcal{O}(R))^\vee$ is an isomorphism, and that \mathfrak{m} is freely generated as a Lie algebra by the spaces V_n . The differential d on \mathfrak{m} is then determined by $d : V_n \rightarrow \mathfrak{m}_{n-1}$, and weight considerations show that the only non-zero contribution is $V_n \rightarrow \prod_{a+b=n-1} [V_a, V_b]$. This is isomorphic to $d : \mathfrak{m}/[\mathfrak{m}, \mathfrak{m}] \rightarrow [\mathfrak{m}, \mathfrak{m}]/[\mathfrak{m}, [\mathfrak{m}, \mathfrak{m}]]$, so must be dual to the cup product.

Therefore, the choice of lift $\mathbb{G}_m \rightarrow \mathrm{Aut}(\mathfrak{m})$ has determined an isomorphism $\mathfrak{m} \cong G\mathrm{H}_{\mathrm{\acute{e}t}}^*(X, \mathcal{O}(R))$. Since this lift was defined in terms of a lift of Frobenius, the isomorphism is equivariant under the outer Frobenius action. \square

Remark 5.11. Under the hypotheses of Corollary 5.9, this allows us to describe the groups $\pi_n^{\mathrm{\acute{e}t}}(X^{\hat{L}}, -) \otimes_{\hat{\mathbb{Z}}} \mathbb{Q}_l$ in terms of cohomology as $H_{n-1}(G(H^*(X, \mathcal{O}(\varpi_f^{\mathrm{\acute{e}t}}(X)^{L, \mathrm{red}}))))$, for G as in Definition 3.25.

5.4. Quasi-formality. Let $j : X \hookrightarrow \bar{X}$ be an open immersion of varieties over \bar{k} , such that locally for the étale topology, the pair (X, \bar{X}) is isomorphic to $(\mathbb{A}^m \times \prod_i (\mathbb{A}^{c_i} - \{0\}), \mathbb{A}^d)$, for some $d = m + \sum c_i$. Note that this includes all geometric fibrations over k in the sense of [Fri] Definition 11.4.

Definition 5.12. For X, \bar{X} as above, let $T = \bar{X} - X$, and let D be the closed subscheme of T of codimension 1 in \bar{X} . Note that $\pi_f^{\mathrm{\acute{e}t}}(X) \rightarrow \pi_f^{\mathrm{\acute{e}t}}(\bar{X} - D)$ is an isomorphism, and define $\pi_f^t(X) := \pi_f^t(\bar{X} - D)$ to be the tame fundamental groupoid (as in [SGA] XIII.2.1.3).

Define $\pi_f^t(X_k)$ similarly, with the tame Weil groupoid $W_f(X_k)$ given by

$$W_f^t(X_k) := \pi_f^t(X_k) \times_{\hat{\mathbb{Z}}} \mathbb{Z}.$$

Let $\varpi_f^t(X) := \pi_f^t(X)$, and define ${}^W\varpi_f^t(X)$ to be the image of $\varpi_f^t(X) \rightarrow W_f^t(X_k)^{\mathrm{alg}}$.

Given a local system \mathbb{V} on X , observe that the direct image $i_*\mathbb{V}$ of \mathbb{V} under the inclusion $i : X \hookrightarrow \bar{X} - D$ is also a local system. We say that \mathbb{V} is tamely ramified along the divisor if $i_*\mathbb{V}$ is tamely ramified along D in the sense of [SGA] Definition XIII.2.1.1.

Lemma 5.13. *Take j as above. If \mathbb{V} is a pure smooth Weil sheaf on Y of weight zero, tamely ramified along the divisor, then $R^\nu j_* \mathbb{V}$ is pure of weight 2ν (in the sense of [KW] Lemma-Definition II.12.7).*

Proof. This is a consequence of the following statements:

- (1) $R^\nu j_* \mathbb{V}$ is pointwise pure of weight 2ν ;
- (2) the canonical map $(R^\nu j_* \mathbb{V})^\vee \rightarrow R\mathcal{H}om_{\bar{X}}(R^\nu j_* \mathbb{V}, \mathbb{Q}_l)$ is an isomorphism.

If $0 \rightarrow \mathbb{V}' \rightarrow \mathbb{V} \rightarrow \mathbb{V}'' \rightarrow 0$ is an exact sequence, with the statements holding for \mathbb{V} and \mathbb{V}'' , then observe that they also hold for \mathbb{V} , since the long exact sequence must degenerate.

The statements are local on \bar{X} . Étale-locally, the pair (X, \bar{X}) is isomorphic to $(U, U') = (\mathbb{A}^m \times \prod_i (\mathbb{A}^{c_i} - \{0\}), \mathbb{A}^d)$, for $d = m + \sum c_i$. We may then reduce to the case when \mathbb{V} is irreducible on U , and so $\mathbb{V} = \mathbb{V}_m \boxtimes \bigotimes_i \mathbb{V}_i$, for \mathbb{V}_i irreducible on $\mathbb{A}^{c_i} - \{0\}$. By the Künneth formula, we now need only consider the pair $(\mathbb{A}^c - \{0\}, \mathbb{A}^c)$.

If \mathbb{V} is constant, then the statements follow from the cohomological purity theorem ([Mil] VI.5.1). Since the scheme $\mathbb{A}^c - \{0\}$ is simply connected for $c > 1$, this leaves only the case $c = 1$. [KW] Lemma I.9.1 shows that $j_* \mathbb{V}$ is pure, and local calculations give $R^i j_* \mathbb{V} = 0$ for $i > 0$ (since \mathbb{V} is tamely ramified, and is non-constant irreducible). \square

Proposition 5.14. *Assume that $j : X_k \hookrightarrow \bar{X}_k$ is a morphism over k , with $j \otimes \bar{k}$ as in Lemma 5.13, for \bar{X}_k proper. If \mathbb{V} is a pure Weil sheaf on X of weight zero, tamely ramified along the divisor, then $H^i(\bar{X}, R^\nu j_* \mathbb{V})$ is pure of weight $i + 2\nu$, for $j : X \rightarrow \bar{X}$ the compactification map.*

Proof. By [Del2], we know that $H^i(\bar{X}, R^\nu j_* \mathbb{V})$ is mixed of weights $\leq i + 2\nu$, since $R^\nu j_* \mathbb{V}$ is pure of weight 2ν . Now, Poincaré duality ([KW] Corollary II.7.3) implies that

$$H^i(\bar{X}, R^\nu j_* \mathbb{V})^\vee \cong H^{2d-i}(\bar{X}, (R^\nu j_* \mathbb{V})^\vee)(2d),$$

which is mixed of weight $\leq -i - 2\nu$, using the isomorphism $(R^\nu j_* \mathbb{V})^\vee \cong R\mathcal{H}om_{\bar{X}}(R^\nu j_* \mathbb{V}, \mathbb{Q}_l)$ of Lemma 5.13. \square

Corollary 5.15. *For X as above, and $\rho : \varpi_f^{\text{ét}} X \rightarrow R$ any Frobenius-equivariant quotient of ${}^W \varpi_f^t(X)^{\text{red}}$, the filtered homotopy type $(X^{\rho, \text{Mal}}, j)$ of Definition 3.43 is quasi-formal (in the sense of Definition 3.30). The formality quasi-isomorphism is equivariant with respect to the outer Frobenius action.*

Proof. This is largely the same as Theorem 5.10. Use the equivalences of Theorem 3.28 to take a filtered minimal model $(\mathfrak{m}, J) \in F\mathcal{S}\hat{\mathcal{N}}(R)$ for $(X^{\rho, \text{Mal}}, j)$. The increasing filtration J_* on \mathfrak{m}^\vee gives a decreasing filtration J^* on \mathfrak{m} , with $J^r \mathfrak{m}_n$ the annihilator of $J_{r-1} \mathfrak{m}^\vee$. Note that $[J^a \mathfrak{m}, J^b \mathfrak{m}] \subset J^{a+b} \mathfrak{m}$ and $J^0 \mathfrak{m} = \mathfrak{m}$.

If we write $\text{Aut}_J(R \ltimes \mathfrak{m}, R)$ for the group of filtered automorphisms of $R \ltimes \mathfrak{m}$ preserving the subgroup R , then similarly to Lemma 3.37, the maps

$$\begin{aligned} & \text{Aut}(R \ltimes \mathfrak{m}, R) \rightarrow \text{Out}_J(X_{\text{ét}}^{\rho, \text{Mal}}) \rightarrow \\ & \{(f, \alpha) : f \in \text{Aut}(R), \alpha \in \text{Iso}_{FDG\text{Alg}(R)}(H_{\text{ét}}^*(\bar{X}, R^* j_* \mathbb{O}(R)), f^\# H_{\text{ét}}^*(\bar{X}, R^* j_* \mathbb{O}(R)))\} \end{aligned}$$

both have pro-unipotent kernels.

We may therefore lift the Frobenius endomorphism $F \in \text{Out}_J(X_{\text{ét}}^{\rho, \text{Mal}})$ to a filtered automorphism of $(R \ltimes \mathfrak{m}, R)$. This gives a lift of the weight decomposition $\mathbb{G}_m \rightarrow R\text{Out}_J(X_{\text{ét}}^{\rho, \text{Mal}})$ to $\text{Aut}_J(\mathfrak{m})$.

Now, $(\mathfrak{m}_n^{\text{ab}})^\vee \cong \bigoplus_{a+b=n+1} H^a(\bar{W}, R^b j_* \mathbb{O}(R)) =: E^{n+1}$, on which J_r is the subspace of weights $\leq n+r+1$. Thus $J^r(\mathfrak{m}_n^{\text{ab}})$ is the subspace of weights $\leq -(n+r+1)$.

The weight restrictions on \mathfrak{m}^{ab} show that $J^r(\text{gr}_\Gamma^s \mathfrak{m})_n = J^r(\text{Lie}_s(\mathfrak{m}^{\text{ab}}))_n$, which is of weights $\leq -(n+r+s)$. This implies that $J^r(\Gamma_s \mathfrak{m})_n$ is of weights $\leq -(n+r+s)$.

We now make a canonical choice of generators by setting

$$\mathcal{W}_{-(n+r+1)} V_n := \mathcal{W}_{-(n+r+1)} J^r \mathfrak{m}_n.$$

Set $V := \prod_i \mathcal{W}_i V$; the weight conditions above show that this has no intersection with $\Gamma_s \mathfrak{m}$ for $s > 1$, so the composition $V \rightarrow \mathfrak{m} \rightarrow \mathfrak{m}^{\text{ab}}$ is injective. Since $\mathcal{W}_{-(n+r+1)}(\mathfrak{m}^{\text{ab}})_n = \mathcal{W}_{-(n+r+1)} J^r(\mathfrak{m}^{\text{ab}})_n$, the composition is also surjective, so V is a space of generators for \mathfrak{m} .

The structure of \mathfrak{m} is now determined by the differentials $d : V_n \rightarrow \mathfrak{m}_{n-1}$. As $\mathfrak{m} = \text{Lie}(V) = V \times \bigwedge^2 V \times \Gamma_3 \mathfrak{m}$, weight and filtration considerations show that we must have the projection $d : V_n \rightarrow (\Gamma_3 \mathfrak{m})_{n-1}$ being 0. The non-zero contributions to d are $V_n \rightarrow V_{n-1}$, which is dual to d_1 on E , and $V_n \rightarrow \prod_{a+b=n-1} [V_a, V_b]$, which must be dual to the cup product. Thus $\mathfrak{m} = G(E)$, as required. \square

Remark 5.16. Under the hypotheses of Corollary 5.9, this allows us to describe the groups $\pi_n^{\text{ét}}(X^{\hat{L}}, -) \otimes_{\mathbb{Z}} \mathbb{Q}_l$ in terms of the Leray spectral sequence as $H_{n-1}(G(\mathcal{J}\mathcal{E}_1^{*,*}))$, where $\mathcal{J}\mathcal{E}_1^{a,b} = H^{2a+b}(\bar{X}, R^{-a} j_* \mathbb{O}(W \varpi_f^{\text{ét}}(X)^{L, \text{red}}))$ as in Definition 3.16, and G as in Definition 3.25.

6. VARIETIES OVER LOCAL FIELDS

6.1. Good reduction, $l \neq p$. Let V be a complete discrete valuation ring, with residue field k (finite, of characteristic $p \neq l$), and fraction field K (of characteristic 0). Let \bar{k}, \bar{K} be the algebraic closures of k, K respectively, and \bar{V} the algebraic closure of V in \bar{K} . As in §4.3, let $\Gamma := \text{Gal}(\bar{K}/K) \times_{\text{Gal}(\bar{k}/k)} \langle F \rangle$.

Let $X_V = \bar{X}_V - T_V$ be a geometric fibration over V (in the sense of [Fri] Definition 11.4). We wish to study the Galois action on the homotopy type $X_{\bar{K}, \text{ét}}$.

Recall from [SGA] Theorem X.2.1 that the map $\pi_f^{\text{ét}}(\bar{X}_k) \rightarrow \pi_f^{\text{ét}}(\bar{X}_V)$ is an equivalence. By *ibid.* §XIII.2.10, this generalises to an equivalence $\pi_f^t(X_k) \rightarrow \pi_f^t(X_V)$. Meanwhile, *ibid.* Corollary XIII.2.8 implies that $\pi_f^t(X_{\bar{K}}) \rightarrow \pi_f^t(X_{\bar{V}})$ is an epimorphism, and *ibid.* Corollary XIII.2.9 shows that $\pi_f^{\text{ét}}(X_{\bar{K}})^{\hat{L}} \rightarrow \pi_f^{\text{ét}}(X_{\bar{V}})^{\hat{L}}$ is an equivalence, where L is any set of prime numbers excluding p .

Proposition 6.1. *If \mathbb{V} is an l -adic local system on $X_{\bar{V}}$, tamely ramified along the divisor (i.e. coming from a representation of $\pi_f^t(X_{\bar{V}})$), then the maps*

$$\begin{aligned} i_\eta^* : H^*(X_{\bar{V}}, \mathbb{V}) &\rightarrow H^*(X_{\bar{K}}, i_\eta^* \mathbb{V}) \\ i_s^* : H^*(X_{\bar{V}}, \mathbb{V}) &\rightarrow H^*(X_{\bar{k}}, i_s^* \mathbb{V}) \end{aligned}$$

are isomorphisms.

Proof. This follows as for [Fri] Theorem 11.5 (which considers only $\pi_f^{\text{ét}}(X_{\bar{V}})^{\hat{L}}$ -representations, but the same proof carries over). \square

Definition 6.2. Since $\pi_1^{\text{ét}}(\text{Spec } V) \cong \text{Gal}(\bar{k}/k)$, we may define $W \varpi_f^t(X_{\bar{V}})$ analogously to Definition 5.2 as the maximal quotient of $\varpi_f^t(X_{\bar{V}}) := \pi_f^t(X_{\bar{V}})^{\text{alg}}$ on which the Frobenius

action is algebraic. Define ${}^W\varpi_f^t(X_{\bar{K}})$ to be the image of $\varpi_f^t(X_{\bar{K}}) \rightarrow {}^W\varpi_f^t(X_{\bar{V}})$, noting that this is a quotient of $\varpi_f^t(X_{\bar{K}})$ on which the Γ -action is potentially unramified.

Theorem 6.3. *Let R be any Frobenius-equivariant reductive quotient of ${}^W\varpi_f^t(X_{\bar{K}})$, with ρ denoting the projection map. Then the outer action of Γ on the homotopy type*

$$X_{\bar{K},\text{ét}}^{\rho,\text{Mal}}$$

is algebraic, potentially unramified (in the sense of §4.2) and mixed (Definition 4.16), giving a canonical Galois-equivariant weight decomposition. It is also quasi-formal, corresponding to the E_1 -term

$$\bigoplus_{a,b} H^a(\bar{X}_{\bar{K}}, R^b j_* \mathbb{O}(R)) \in \text{FDGAlg}(R),$$

of the Leray spectral sequence for the immersion $j : X \rightarrow \bar{X}$. The formality quasi-isomorphism is equivariant with respect to the outer Galois action.

Proof. We know that the homotopy type is given by

$$C_{\text{ét}}^\bullet(X_{\bar{K}}, \mathbb{O}(R)) \in c\text{Alg}(R).$$

From the definition of ${}^W\varpi_f^t(X_{\bar{K}})$, we know that $\mathbb{O}(R)$ is the pullback of a local system on $X_{\bar{V}}$, so $i_{\eta*} \mathbb{O}(R)$ is a local system and $i_{\eta}^* i_{\eta*} \mathbb{O}(R) = \mathbb{O}(R)$.

The equivalences of Proposition 6.1 now give quasi-isomorphisms

$$C_{\text{ét}}^\bullet(X_{\bar{K}}, \mathbb{O}(R)) = C_{\text{ét}}^\bullet(X_{\bar{K}}, i_{\eta}^* i_{\eta*} \mathbb{O}(R)) \leftarrow C_{\text{ét}}^\bullet(X_{\bar{V}}, i_{\eta*} \mathbb{O}(R)) \rightarrow C_{\text{ét}}^\bullet(X_{\bar{V}}, i_s^* i_{\eta*} \mathbb{O}(R)).$$

To show that this is a potentially unramified representation, restrict attention to the subgroupoid of R on K' -valued points, noting that this is unramified for $\text{Gal}(\bar{K}/K')$. We may now adapt Proposition 5.6 to see that this has a canonical weight decomposition, and Corollary 5.15 to see that this is quasi-formal. The precise expression of quasi-formality is obtained by noting that all of the quasi-isomorphisms above extend naturally to the filtered algebras of Corollary 5.15. \square

Corollary 6.4. *If L is a set of primes including l , and:*

- (1) $\pi_f^{\text{ét}}(X)^{\hat{L}}$ *is algebraically good relative to* ${}^W\varpi_f^t(X_{\bar{K}})^{\hat{L}}$,
- (2) $\pi_n^{\text{ét}}(X^{\hat{L}}) \otimes_{\mathbb{Z}} \mathbb{Q}_l$ *is finite-dimensional for all* $n > 1$, *and*
- (3) *the action of* $\ker(\pi_f^{\text{ét}}(X_{\bar{K}})^{\hat{L}} \rightarrow \pi_f^t(X_{\bar{V}})^{\hat{L}})$ *on* $\pi_n^{\text{ét}}(X^{\hat{L}}) \otimes_{\mathbb{Z}} \mathbb{Q}_l$ *is unipotent for all* $n > 1$,

then the Galois action on $\pi_n^{\text{ét}}(X^{\hat{L}}) \otimes_{\mathbb{Z}} \mathbb{Q}_l$ *is potentially unramified and mixed, giving it a canonical weight decomposition. It may also be recovered from the Leray spectral sequence, as in Remark 5.16.*

Proof. Substitute $R = \pi_f^t(X_{\bar{V}})^{L,\text{red}}$ into Theorem 6.3 and Remark 5.16. \square

Remark 6.5. Note that if L does not contain p , then the third condition of the Corollary is vacuous.

6.2. Good reduction, $l = p$. Let X, \bar{X}, V, K, k etc. be as in the previous section. Let $W = W(k)$, the ring of Witt vectors over k , and K_0 the fraction field of W ; let $W^{\text{nr}} := W(\bar{k})$, with K_0^{nr} its fraction field. Write $\Gamma = \text{Gal}(\bar{K}/K)$, and choose a homomorphism $\sigma : K \rightarrow K$ extending the natural action of the Frobenius operator F on $W(k) \subset K$. Let \tilde{X}/W be a smooth formal scheme lifting X_k .

Assume moreover that $X_V = \bar{X}_V - D_V$, for D_V a divisor of simple normal crossings.

Definition 6.6. Similarly to Definition 5.2, define ${}^{\text{Gal}}\varpi_f^{\text{ét}}(\bar{X}_{\bar{V}})$ to be the image of $\varpi_f^{\text{ét}}(\bar{X}_{\bar{V}}) \rightarrow \varpi_f^{\text{ét}}(\bar{X}_V)$. Representations of ${}^{\text{Gal}}\varpi_f^{\text{ét}}(\bar{X}_{\bar{V}})$ correspond to \mathbb{Q}_l -local systems on $\bar{X}_{\bar{V}}$ arising as subrepresentations of pullbacks of local systems on \bar{X}_V . Note that the action of $\text{Gal}(\bar{K}/K)$ on ${}^{\text{Gal}}\varpi_f^{\text{ét}}(\bar{X}_{\bar{V}})$ is algebraic and potentially unramified.

Define ${}^{\text{pnr}}\varpi_f^{\text{ét}}(X_{\bar{K}})$ to be the image of $\varpi_f^{\text{ét}}(X_{\bar{K}}) \rightarrow {}^{\text{Gal}}\varpi_f^{\text{ét}}(\bar{X}_{\bar{V}})$, noting that this is equipped with a potentially unramified algebraic action of $\text{Gal}(\bar{K}/K)$.

Lemma 6.7. *The category of representations of $\text{Gal}(\bar{k}/k)^{\text{alg},0} \ltimes {}^{\text{Gal}}\varpi_f^{\text{ét}}(\bar{X}_{\bar{V}})$ is equivalent to the category of unit-root F -lattices on $\mathfrak{X} \otimes_W W^{\text{nr}}/W^{\text{nr}}$.*

Proof. First observe that the latter category is the direct limit over finite extensions k'/k of the categories of unit-root F -lattices on $\mathfrak{X} \otimes_W W(k')/W(k')$. Note that $\text{Gal}(\bar{k}/k)^{\text{alg},0} \ltimes {}^{\text{Gal}}\varpi_f^{\text{ét}}(\bar{X}_{\bar{V}}) \simeq \varprojlim {}^{\text{Gal}}\varpi_f^{\text{ét}}(\bar{X}_{k'})$. By [Kat] 4.1.1, $\text{Rep}(\varpi_f^{\text{ét}}(\bar{X}_{k'}))$ is equivalent to the category of unit-root F -lattices on $\mathfrak{X} \otimes_W W(k')/W(k')$. Under this equivalence, a local system \mathbb{V} on $X_{k'}$ will correspond to $\mathbb{V} \otimes_{\mathbb{Q}_p} \mathcal{O}_{\mathfrak{X} \otimes_W W^{\text{nr}}}$. \square

Definition 6.8. Given a Galois-equivariant quotient R of ${}^{\text{pnr}}\varpi_f^{\text{ét}}(X_{\bar{K}})$, $\mathbb{O}(R)$ is an R -representation in \mathbb{Q}_p -local systems on $X_{\bar{K}}$. Similarly to Theorem 6.3, $i_{\eta*}\mathbb{O}(R)$ is a local system on $X_{\bar{V}}$. Now, we have $i_s^*i_{\eta*}\mathbb{O}(R)$ a local system on $X_{\bar{k}}$, equipped with an action of $\text{Gal}(\bar{K}/K)$ compatible with the action on $X_{\bar{k}}$.

Since this action is potentially unramified, it gives us an action of $\text{Gal}(\bar{k}/k)^{\text{alg},0}$ on $i_s^*i_{\eta*}\mathbb{O}(R)$, and we may thus define a unit-root F -isocrystal

$$\mathcal{O}(R) := i_s^*i_{\eta*}\mathbb{O}(R) \otimes_{\mathbb{Q}_p} \mathcal{O}_{\mathfrak{X} \otimes_W W^{\text{nr}}}$$

on $\mathfrak{X} \otimes_W W^{\text{nr}}/W^{\text{nr}}$.

From now on, let $B = B_{\text{cris}}$, as defined in [Fal].

Lemma 6.9. *For R as above, there is a canonical (ϕ, \mathcal{G}^0) -equivariant weak equivalence in $\text{Ho}(c\text{Alg}_B(R))$:*

$$\mathbf{C}_{\text{ét}}^{\bullet}(X_{\bar{K}}, \mathbb{O}(R)) \otimes_{\mathbb{Q}_p} B \sim \mathbf{C}_{\text{cris}}^{\bullet}(X_{\bar{k}}/W^{\text{nr}}, \mathcal{O}(R)) \otimes_{K_0^{\text{nr}}} B,$$

where $\mathbf{C}_{\text{cris}}^{\bullet}(-)$ is defined as in [Pri1] p.14 and p.17, corresponding to $\mathbb{R}\Gamma_{\text{cris}}(-)_{\mathbf{U}}$ in [Ols] 3.25.

Proof. This equivalence is given by adapting [Ols] Theorem 1.6 to the category $\mathcal{C} := \text{Rep}(R)$, using the comparison of Remark 2.38.

Another approach to proving this would be to replace $\mathbb{Q}_p, \mathcal{O}_X$ by $\mathbb{O}(R), \mathcal{O}(R)$ in [Pri1] §§3.1, 3.2. Alternatively, [Fal] proves that there is an isomorphism on the corresponding cohomology groups, respecting cup products, since $\mathbb{O}(R)$ and $\mathcal{O}(R)$ are “associated”. With a little more care, this could be extended to a quasi-isomorphism of the minimal E_{∞} -algebras they underlie. Remark 3.7 then implies that the corresponding objects in $dg\hat{\mathcal{N}}(R)$ are weakly equivalent. \square

In fact, we may extend this to a filtered version:

Definition 6.10. For a topos \mathcal{T} , if $\mathcal{C}_{\mathcal{T}}^{\bullet}(\mathcal{S})$ is a canonical cosimplicial \mathcal{T} -resolution of a sheaf \mathcal{S} of algebras on X , with $\mathbf{C}_{\mathcal{T}}^{\bullet}(X, \mathcal{S}) := \Gamma(X, \mathcal{C}_{\mathcal{T}}^{\bullet}(\mathcal{S}))$, then for any morphism $f : X \rightarrow Y$ we have a bicosimplicial algebra $\mathbf{C}_{\mathcal{T}}^{\bullet}(Y, f_*\mathcal{C}_{\mathcal{T}}^{\bullet}(\mathcal{S}))$, and we define

$$\mathbf{C}_{\mathcal{T}}^{\bullet}(f, \mathcal{S}) := \tau''\mathbf{C}_{\mathcal{T}}^{\bullet}(Y, f_*\mathcal{C}_{\mathcal{T}}^{\bullet}(\mathcal{S})) \in Fc\text{Alg},$$

defined as in Definition 3.39.

Lemma 6.11. *For R as above and $j : X \rightarrow \bar{X}$, there is a canonical (ϕ, \mathcal{G}^0) -equivariant weak equivalence in $\mathrm{Ho}(\mathrm{FcAlg}_B(R))$:*

$$\mathbf{C}_{\text{ét}}^\bullet(j_{\bar{K}}, \mathcal{O}(R)) \otimes_{\mathbb{Q}_p} B \sim \mathbf{C}_{\text{cris}}^\bullet(j_k/W^{\text{nr}}, \mathcal{O}(R)) \otimes_{K_0^{\text{nr}}} B,$$

in $\mathrm{FcAlg}(R)$.

Proof. The proof of Lemma 6.9 adapts. \square

Definition 6.12. For R as above, define the log-crystalline homotopy type $X_{\text{cris}}^{\rho, \text{Mal}} \in \mathrm{Ho}(\mathrm{sAGpd}_{K_0^{\text{nr}}})$ of X over K_0^{nr} by

$$X_{\text{cris}}^{\rho, \text{Mal}} := (R \otimes_{\mathbb{Q}_p} K_0^{\text{nr}}) \ltimes G(\mathbf{C}_{\text{cris}}^\bullet(X_k/W^{\text{nr}}, \mathcal{O}(R))).$$

The above lemma thus gives a \mathcal{G}^0 -equivariant quasi-isomorphism $X_{\text{cris}}^{\rho, \text{Mal}} \otimes_{K_0^{\text{nr}}} B \sim X_{\bar{K}, \text{ét}}^{\rho, \text{Mal}} \otimes_{\mathbb{Q}_p} B$ of filtered homotopy types.

Theorem 6.13. *Given a Galois-equivariant quotient R of ${}^{\text{pnr}}\varpi_f^{\text{ét}}(X_{\bar{K}})$, with quotient map ρ , the outer Galois action on $X_{\bar{K}, \text{ét}}^{\rho, \text{Mal}}$ is algebraic and potentially crystalline.*

Proof. In the notation of §4.3, we need to show that the map $\mathcal{G} \rightarrow \mathrm{Out}(X_{\bar{K}, \text{ét}}^{\rho, \text{Mal}})$ factors through $\mathcal{G}^{\text{pcris}}$. Apply Corollary 4.21 to Lemma 6.9, taking

$$Y = \mathrm{Out}(X_{\bar{K}, \text{ét}}^{\rho, \text{Mal}}) \times_{\mathrm{Aut}(R)} \mathcal{G}^{\text{pnr}, 0}, \quad Z = \mathrm{Out}(X_{\text{cris}}^{\rho, \text{Mal}}) \times_{\mathrm{Aut}(R \otimes_{\mathbb{Q}_p} K_0^{\text{nr}})} (\mathbb{Z}^{\text{alg}, 0} \otimes_{\mathbb{Q}_p} K_0^{\text{nr}}),$$

with the \mathcal{G}^0 action on $\mathcal{G}^{\text{pnr}, 0}$ given by multiplication, and with Z having trivial \mathcal{G}^0 -action.

Since this action on \mathcal{G}^{pnr} is potentially unramified, we have $\mathcal{G}^{\text{pnr}, 0} \otimes_{\mathbb{Q}_p} B \cong (\mathbb{Z}^{\text{alg}, 0} \otimes_{\mathbb{Q}_p} K_0^{\text{nr}}) \otimes_{K_0^{\text{nr}}} B$ with the Frobenius action on $(\mathbb{Z}^{\text{alg}, 0} \otimes_{\mathbb{Q}_p} K_0^{\text{nr}})$ coming from the identification $\mathbb{Z} \cong \langle F \rangle$, since $\mathcal{G}^{\text{pnr}, 0} \cong \mathbb{Z}^{\text{alg}, 0}$.

Given a B -algebra A , and fixing a point in $\mathcal{G}^{\text{pnr}, 0}(A)$, with image $\theta^{\text{red}} \in \mathrm{Aut}(R)(A)$, it then suffices to show that there are functorial \mathcal{G}^0 -equivariant isomorphisms

$$\begin{aligned} & \mathrm{Iso}_{\mathrm{Ho}(\mathrm{dgAff}_A(R))}(X_{\bar{K}, \text{ét}}^{\rho, \text{Mal}} \otimes_{\mathbb{Q}_p} A, (\theta^{\text{red}})^\# X_{\bar{K}, \text{ét}}^{\rho, \text{Mal}} \otimes_{\mathbb{Q}_p} A) \\ & \cong \mathrm{Iso}_{\mathrm{Ho}(\mathrm{dgAff}_A(R))}(X_{\text{cris}}^{\rho, \text{Mal}} \otimes_{K_0^{\text{nr}}} A, (\theta^{\text{red}})^\# X_{\text{cris}}^{\rho, \text{Mal}} \otimes_{K_0^{\text{nr}}} A), \end{aligned}$$

which follow from the Galois-equivariant isomorphism $X_{\bar{K}, \text{ét}}^{\rho, \text{Mal}} \otimes_{\mathbb{Q}_p} B \cong X_{\text{cris}}^{\rho, \text{Mal}} \otimes_{K_0^{\text{nr}}} B$ of Lemma 6.9. \square

Similarly to the description in Definition 5.2, we have:

Lemma 6.14. *Fix an embedding $\iota : \mathbb{Q}_p \rightarrow \mathbb{C}$, and lift Frobenius to $\sigma \in \mathrm{Gal}(\bar{K}/K)$. The σ action on a Galois-equivariant quotient R of ${}^{\text{pnr}}\varpi_f^{\text{ét}}(\bar{X}_{\bar{K}})$ is ι -pure of weight zero if and only if, for all R -representations V , the corresponding local systems \mathbb{V} on $\bar{X}_{\bar{K}}$ are all subsystems of pullbacks of ι -pure local systems on \bar{X}_k . We then say that R is ι -pure.*

Theorem 6.15. *Given an ι -pure Galois-equivariant quotient R of ${}^{\text{pnr}}\varpi_f^{\text{ét}}(\bar{X}_{\bar{K}})$, with quotient map ρ , the outer Galois action on $X_{\bar{K}, \text{ét}}^{\rho, \text{Mal}}$ is ι -mixed in the sense of Definition 4.23, giving a canonical weight decomposition on $X_{\bar{K}, \text{ét}}^{\rho, \text{Mal}} \otimes B^\sigma$.*

Proof. This is essentially the same as Proposition 5.6. Frobenius gives a canonical action $\mathbb{Z}^{\text{alg}, 0} \rightarrow \mathrm{Out}(X_{\text{cris}}^{\rho, \text{Mal}})$. It will suffice to show that this is ι -mixed of integral weights. By Lemma 3.14, we need only consider the Frobenius action on cohomology

$$H_{\log \text{cris}}^*(X_k/W^{\text{nr}}, \mathcal{O}(R)).$$

The Leray spectral sequence gives

$$H_{\text{cris}}^{2a+b}(\bar{X}_k/W^{\text{nr}}, R_{\log \text{cris}}^{-a} j_* \mathcal{O}(R)) \implies H^{a+b}(X, \mathcal{O}(R)).$$

If we write $D^{(1)}$ for the normalisation of D , $D^{(n)}$ for its n -fold intersection, and $i_n : D^{(n)} \rightarrow \bar{X}$ for the embedding, then as in [Del1] 3.2.4.1, there is an isomorphism

$$H_{\text{cris}}^{2a+b}(\bar{X}_k/W^{\text{nr}}, R_{\text{cris}}^{-a} j_* \mathcal{O}(R)) \cong H_{\text{cris}}^{2a+b}(D_k^{(-a)}, i_n^* j_* \mathcal{O}(R)(a)),$$

since $j_* \mathcal{O}(R)$ is associated to a locally constant sheaf on \bar{X} .

Now, [Ked] Theorem 6.6.2 combined with Poincaré duality proves that $H_{\text{cris}}^{2a+b}(D_k^{(-a)}, i_n^* j_* \mathcal{O}(R)(a))$ is ι -pure of weight b . \square

Theorem 6.16. *For ρ as above, $X_{\bar{K}, \text{ét}}^{\rho, \text{Mal}}$ is quasi-formal, corresponding to the E_1 -term*

$$jE_1^{a,b}(X_{\bar{K}, \text{ét}}^{\rho, \text{Mal}}) = \bigoplus_{a,b} H^{2a+b}(\bar{X}_{\bar{K}}, R^{-b} j_* \mathbb{O}(R)) \in FDG\text{Alg}(R),$$

of the Leray spectral sequence for the immersion $j : X \rightarrow \bar{X}$. The formality quasi-isomorphism on $X_{\bar{K}, \text{ét}}^{\rho, \text{Mal}} \otimes B^\sigma$ can be chosen to be equivariant with respect to the outer Galois action.

Proof. Since the Galois action is ι -mixed in the sense of Definition 4.23, there is a Galois-equivariant weight decomposition $\mathbb{G}_m \rightarrow \text{ROut}_J(X_{\bar{K}, \text{ét}}^{\rho, \text{Mal}} \otimes B^\sigma)$, using Lemma 4.22. The argument of Corollary 5.15 now adapts to show that $X_{\bar{K}, \text{ét}}^{\rho, \text{Mal}} \otimes B^\sigma$ is quasi-formal, with the formality quasi-isomorphism equivariant under the Galois action.

In particular this implies that

$$\text{ROut}_J(X_{\bar{K}, \text{ét}}^{\rho, \text{Mal}})(B^\sigma) \rightarrow \text{Aut}(jE_1^{*,*}(X_{\bar{K}}^{\rho, \text{Mal}}))(B^\sigma)$$

is surjective. Thus the corresponding morphism of pro-algebraic groups is surjective, allowing us to lift the weight decomposition on $E_1^{*,*}(X_{\bar{K}}^{\rho, \text{Mal}})$ non-canonically to $X_{\bar{K}, \text{ét}}^{\rho, \text{Mal}}$. This decomposition need not be compatible with the canonical decomposition on $X_{\bar{K}, \text{ét}}^{\rho, \text{Mal}} \otimes B^\sigma$.

The argument of Corollary 5.15 adapted to this decomposition now shows that $X_{\bar{K}, \text{ét}}^{\rho, \text{Mal}}$ is quasi-formal. \square

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